Eighth exercise sheet Advanced Algebra II. In the following it can be taken for granted that a subgroup Θ of an ordered abelian group Γ is convex if and only if $[0,\theta]_\Gamma \subseteq \Theta$ for all $\theta \in \Theta$. It can also be taken for granted that the convex hull of a subring of an ordered field is a subring of that field.

Problem 1 (6 points). *Let K be a field, Γ an ordered abelian group and $K^\times \xrightarrow{v} \Gamma$ a surjective group homomorphism defining a valuation of K and $R = \{k \in K \mid v(k) \geq 0\}$ the valuation ring defined by v. Show that we have a bijection between* $\mathrm{Spec}R$ *and the set of convex subgroups $\Theta \subseteq \Gamma$, sending $\mathfrak{p} \in \mathrm{Spec}R$ to $\Theta = v^{-1}R_\mathfrak{p}^\times$ and Θ to $\mathfrak{p} = \{r \in R \mid v(r) \notin \Theta\}$.*

Problem 2 (2 points). *In the situation of the previous problem, describe the topological space* $\mathrm{Spec}R$ *in terms of the ordered abelian group Θ!*

Problem 3 (2 points). *In the situation of the first problem, let $(k_i)_{i=1}^n \in K^n$ with $n \geq 2$ such that $\sum_{i=1}^n k_i = 0$. Show that $\min_{1\leq i\leq n} v(a_i)$ is attained at least twice!*

Problem 4 (2 points). *In the situation of the first problem, let $K_o \subseteq K$ be a subfield over which K is algebraic. For all $k \in K^\times$, show that there exists a positive integer n such that $nv(k) \in v(L^\times)$!*

Problem 5 (2 points). *In the situation of the previous problem, let $R_o = R \cap K_o$. Show that* $\mathrm{Spec}R_o \to \mathrm{Spec}R$ *is bijective!*

Problem 6 (2 points). *In the situation of the previous problem, show that* $\mathrm{Spec}R \to \mathrm{Spec}R_o$ *is in fact a homeomorphism!*

Problem 7 (6 points). *Let K_o be an ordered field and K its real closure. Show that we have a bijection between the sets of convex subrings $R \subseteq K_o$ and $S \subseteq K$, sending R to its convex hull in K and S to $S \cap K_o$.*

Let $m \in \mathbb{N}$ and $\preceq$ the partial ordering of $\mathbb{N}^m$ by $\alpha \preceq \beta$ iff $\alpha_i \leq \beta_i$ for $1 \leq i \leq m$.

Problem 8 (6 points). *Let $\trianglelefteq$ be a linear order on $\mathbb{N}^n$ which is translation invariant in the sense that $\alpha \trianglelefteq \beta$ implies $\alpha + \gamma \trianglelefteq \beta + \gamma$ for all $\gamma \in \mathbb{N}^n$. Show that the following conditions are equivalent:*

- *$\trianglelefteq$ extends $\preceq$.*
- *$\trianglelefteq$ is a well-ordering of $\mathbb{N}^n$.*
- *$\mathbb{N}^n$ has a $\trianglelefteq$-smallest element.*
- *0 is the $\trianglelefteq$-smallest element of $\mathbb{N}^n$.*

Eight of the 28 points for this sheet are bonus points. Solutions should be submitted in the lecture Friday, June 14.