Seventh exercise sheet Advanced Algebra II. A *splitting* of an open covering $X = \bigcup_{i \in I} U_i$ of a topological space X is a locally constant map $X \xrightarrow{\iota} I$ such that $x \in U_{\iota(x)}$ for all $x \in X$.

Problem 1 (2 points)**.** *Show that every open covering of a spectral Hausdorff space admits a splitting.*

Recall from the lecture in the previous term that for a spectral space X, the connected components (the maximal connected subsets) and the quasi-components (the minimal non-empty intersections of clopen subsets) coincide and that the set $\pi_0 X$ of connected components of X is a compact spectral space when equipped with the quotient topology, and the map $X \xrightarrow{\pi_X} \pi_0 X$ spectral. It is clear from this description that the connected components are non-empty closed subsets of X. By the general properties of spectral spaces, each of them contains at least one closed point.

For a spectral space X, let X_c be the set of closed points of X. As this is closed under specialization it follows from Proposition 2.3.1 of the lecture that X_c is closed in X if and only if it is closed in X^{con}.

Problem 2 (9 points)**.** *Let X be a spectral space such that X_c is closed in X. Show that the following conditions are equivalent:*

- *X has closed stars.*
- *Every element k of $\pi_0 X$ contains but one closed point $c(k)$ of X.*
- *Every open covering of X has a splitting.*

If this holds, show that the map $\pi_0 X \xrightarrow{c} X$ is spectral and the map $X_c \xrightarrow{\pi_X} \pi_0 X$ a homeomorphism.

Let $\mathfrak{k}$ be a field, $A = \mathfrak{k}[T]$, $S = \mathrm{Sper}\mathfrak{k}$, $X = \mathrm{Sper}A$, $X \xrightarrow{f} S$ the map induced by the embedding $\mathfrak{k} \to A$.

Problem 3 (2 points)**.** *Let $U = X$ or $U = \mathcal{P}(T-a)$ or $U = \mathcal{P}(a-T)$ for some $a \in \mathfrak{k}$. Show that*

$$R^p(f|_U)_* \underline{\mathbb{Z}}_U \cong \begin{cases} \underline{\mathbb{Z}}_S & p = 0 \\ 0 & p > 0 \end{cases}$$

Problem 4 (3 points)**.** *Let $U = \mathcal{P}(T-a) \cap \mathcal{P}(b-T) \subseteq X$ and $V = \mathcal{P}(b-a) \subseteq S$. Let $V \xrightarrow{j} S$ denote the inclusion. Show that*

$$R^p(f|_U)_* \underline{\mathbb{Z}}_U \cong \begin{cases} j_* \underline{\mathbb{Z}}_V & p = 0 \\ 0 & p > 0 \end{cases}$$

Problem 5 (4 points)**.** *Let K be a field and $R \subseteq K$ a valuation ring of K. Show that we have a bijection between* $\mathrm{Spec}R$ *and the set of subrings $S \subseteq K$ containing R, sending $\mathfrak{p} \in \mathrm{Spec}R$ to $S = R_{\mathfrak{p}}$. Also show that every such S is a valuation ring and that the inverse map sends S to $\mathfrak{p} = \mathfrak{m}_S \cap R$.*

Solutions should be submitted in the lecture Friday, June 7.