Fifth exercise sheet Advanced Algebra II.

Problem 1 (7 points). *Let $X \xrightarrow{f} Y$ be a continuous map and $y \in Y$ such that the following assumptions hold:*

- *The fibre F of f at y is finite: $f^{-1}\{y\} = \{x_1, \dots, x_m\} =: F$.*
- *For $1 \leq i < j \leq m$, x_i and x_j have disjoint injective object of $\mathcal{A}$ belongs to $\mathcal{X}$.*
- *If $A \subseteq X$ is closed and $A \cap F = \emptyset$ then $y \notin \overline{f(A)}$ where the closure is taken in Y.*

For a sheaf $\mathcal{G}$ of sets on X, show that we have an isomorphism

$$\left(f_*\mathcal{G}\right)_y \xrightarrow{\cong} \prod_{i=1}^{m} \mathcal{G}_{x_i}$$

given by the collection of the morphisms (1.4.2) from the lecture!

Problem 2 (2 points). *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, where $\mathcal{A}$ has sufficiently many injective objects. Let $0 \to F' \to F \to F'' \to 0$ be a sequence of functors from $\mathcal{A}$ to $\mathcal{B}$ such that $0 \to F'X \to FX \to F''X$ is exact for arbitrary objects X of A, and such that $FX \to F''X$ is an epimorphism when X is injective. Construct a long exact sequence*

$$0 \to F'X \to FX \to F''X \to R^1F'X \to \dots$$
$$\dots \to R^{p-1}F''X \to R^pF'X \to R^pFX \to R^pF''X \to R^{p+1}F'X \to \dots$$

Problem 3 (3 points). *Let X be an arbitray topological space which the union of its open subsets U and V. Construct an exact sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to \dots$$
$$\to H^{p-1}(U \cap V, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(U, \mathcal{F}) \oplus H^p(V, \mathcal{F}) \to H^p(U \cap V, \mathcal{F}) \to \dots$$

A point x of a spectral space X will be called constructible if $\{x\}$ is a constructible subset of X.

Problem 4 (8 points). *Let X be a spectral space and $\mathfrak{B}_o$ a topology base on X and $\mathfrak{B}_c$ a subset of*

$$\mathfrak{B}_c = \left\{ \{x\} \;\middle|\; x \text{ is a construcible closed point of } X. \right\}$$

such that the elements of $\mathfrak{B}_o$ are quasi-compact, the intersection of two elements of $\mathfrak{B}_o$ is a (possibly empty) disjoint union of elements of $\mathfrak{B}_o$ and such that for $\Omega \in \mathfrak{B}_o$, $X \setminus \Omega$ is a (possibly empty) finite disjoint union of elements of $\mathfrak{B}$, where $\mathfrak{B} = \mathfrak{B}_o \cup \mathfrak{B}_c$.

Show the following:

- *$\mathfrak{B}$ is a topology base of X^{con}.*

- *Every open covering of a clopen subset of* X^{con} *can be refined to a finite disjoint decomposition into elements of* $\mathfrak{B}_c$.
- *For every constructible closed point* $x \in X$ *we have* $\{x\} \in \mathfrak{B}$.
- *If* $x \neq y$ *and* x *is a specialization of* y*, then* x *is a constructible closed point of* X.
- *The Krull dimension of* X *is at most one.*

Solutions should be submitted in the lecture Friday, May 17.