Second exercise sheet Advanced Algebra II.

Problem 1 (4 points). *Let $\mathcal{F} \xrightarrow{f} \mathcal{G}$ be a morphism of presheaves of sets on the topological space X, where $\mathcal{F}$ is separated. Show that the map between stalks $\mathcal{F}_x \to \mathcal{G}_x$ induced by f is injective for all $x \in X$ if and only if the map $\mathcal{F}(U) \xrightarrow{f} \mathcal{G}(U)$ is injective for all open subsets $U \subseteq X$.*

Problem 2 (4 points). *Let $\mathcal{F} \xrightarrow{f} \mathcal{G}$ be a morphism of sheaves of sets on the topological space X. Show that the map between stalks $\mathcal{F}_x \to \mathcal{G}_x$ induced by f is bijective for all $x \in X$ if and only if the map $\mathcal{F}(U) \xrightarrow{f} \mathcal{G}(U)$ is bijective for all open subsets $U \subseteq X$.*

Let $\mathcal{F}$ be a presheaf of sets on X and $\big(\mathrm{Sheaf}(\mathcal{F})\big)(U)$ the set of all $(f_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ such that the open subsets $V \subseteq U$ with the property that there is $\phi \in \mathcal{F}(V)$ such that

$$f_x = (\text{image of } \phi \text{ under } \mathcal{F}(V) \to \mathcal{F}_x)$$

holds for all $x \in V$ cover X. With the obvious maps of restriction to smaller open subsets, this is a presheaf of sets of X, and we have a morphism

$$\mathcal{F} \xrightarrow{\iota_{\mathcal{F}}} \mathrm{Sheaf}(\mathcal{F}) \tag{1}$$

defined by

$$\iota_{\mathcal{F}}(\phi) = \big(\text{image of } \phi \text{ under } \mathcal{F}(U) \to \mathcal{F}_x\big)_{x\in U}$$

for $\phi \in \mathcal{F}(U)$.

Problem 3 (3 points). *Show that the presheaf* $\mathrm{Sheaf}(\mathcal{F})$ *is a sheaf.*

It is easy to see that we have a well-defined morphism

$$\mathrm{Sheaf}(\mathcal{F})_\xi \xrightarrow{\kappa_{\mathcal{F},\xi}} \mathcal{F}_\xi\mathcal{F}_\xi \tag{2}$$

sending, for arbitrary open neighbourhoods U of ξ, the image in $\mathrm{Sheaf}(\mathcal{F})_\xi$ of $(f_x)_{x\in U} \in \mathrm{Sheaf}\big(\mathcal{F}(U)\big)$ to f_ξ.

Problem 4 (4 points). *For an arbitrary presheaf of sets $\mathcal{F}$, show that $\kappa_{\mathcal{F},\xi}$ and the morphism induced by $\iota_{\mathcal{F}}$ on stalks at ξ are inverse to each other bijections $\mathcal{F}_\xi \cong \mathrm{Sheaf}(\mathcal{F})_\xi$.*

Problem 5 (2 points). *Let $\mathcal{F}$ be a presheaf of sets on the topological space X. Then the morphism $\mathcal{F} \to \mathrm{Sheaf}(\mathcal{F})$ is an isomorphism if and only if $\mathcal{F}$ is a sheaf.*

Let $(\mathcal{F}_\lambda)_{\lambda\in\Lambda}$ be a family of sheaves of abelian groups on X. Let $\big(\coprod_{\lambda\in\Lambda}\mathcal{F}_\lambda\big)(U)$ be the set of all $(f_\lambda)_{\lambda\in\Lambda}\mathcal{F}_\lambda(U)$ such that the open subsets $V\subseteq U$ with the property that there are only finitely many $\lambda\in\Lambda$ with $f_\lambda|_V\neq 0$.

Problem 6 (3 points)**.** *Show that this a coproduct of the $\mathcal{F}_\lambda$ in the category of sheaves, and calculate its stalk!*

Solutions should be submitted in the lecture Friday, April 26.