Eleventh exercise sheet Advanced Algebra II.

Problem 1 (4 points). *Let* $\mathfrak{P} \in \mathrm{Sper} A$, $\mathfrak{p} = \mathrm{supp}\mathfrak{P}$ *and let* $\mathfrak{q} \in \mathrm{Spec} A$ *such that that* $\mathfrak{q}$ *is* $\mathfrak{P}$*-convex and* $\mathfrak{p}$ *a proper subset of* $\mathfrak{q}$*. Show that the neigbourhoods*

$$\mathcal{P}(a_1, \dots, a_n) \qquad\qquad a_i \in A \setminus \big((-\mathfrak{P}) \cup \mathfrak{q}\big) \text{ for } 1 \le i \le n$$

do not form a neighbourhood base of $\mathfrak{P}$ *in* $\mathrm{Sper} A$.

Problem 2 (2 points). *Let* $A \xrightarrow{\beta} B$ *be a ring homomorphism such that* $\mathrm{Sper}\beta$ *induces a homeomorphism of* $\mathrm{Sper} B$ *with a constructible open subset* $U \subseteq \mathrm{Sper} A$*. Show that a closed subset of* $\mathrm{Sper} B$ *is* A*-proper if and only if its image in* U *is a closed subset of* A.

Problem 3 (4 points). *Let* X *be a spectral space such that* X^{inv} *has closed stars, and let* X^{Ber} *be the Berkovich-Quotient of* X*. For* $\xi \in X^{\mathrm{Ber}}$*, let* $X_\xi \subseteq X$ *be its inverse image under* $\mathbb{B}_X$*. Show that the base-change homomorphism*

$$\Big(R^* \mathbb{B}_{X*} \mathcal{F}\Big)_\xi \to H^*(X_\xi, \mathcal{F})$$

is in isomorphism!

Problem 4 (2 points). *Let* Y *be an irreducible spectral space. Show that a sheaf on* Y *is constant if and only if its cospecialization homomorphisms are isomorphisms!*

Problem 5 (4 points). *In the situation of the third problem, show that* $\mathbb{B}_X^*$ *is an equivalence of the category of sheaves on* X^{Ber} *with the category of sheaves* $\mathcal{F}$ *on* X *for which all cospecialization homomorphisms are isomorphisms, with an inverse functor given by* $\mathbb{B}_{X*}$*! Also, show that for all such* $\mathcal{F}$ *and all* $p > 0$ $R^p \mathbb{B}_{X*} \mathcal{F} = 0$*!*

Let $\mathfrak{l}/\mathfrak{k}$ be an extension of real closed fields which has transcendence degree 1. Let $\ell \in \mathfrak{l}$ such that both ℓ and ℓ^{-1} are elements of the convex hull of $\mathfrak{k}$ in $\mathfrak{l}$. Let $A = \mathfrak{k}[X, Y]$, K its field of quotients and let $\mathfrak{P}$ be the cone of all $f \in A$ such that $f(\varepsilon, \ell\varepsilon) \ge 0$ for all sufficiently small non-negative $\varepsilon \in \mathfrak{k}$.

Problem 6 (6 points). *In this situation, show that* $\mathrm{supp}\mathfrak{P} = \{0\}$*, that the convex hull of* A *in* K *equipped with the ordering defined by* $\mathfrak{P}$ *is a discrete valuation ring with residue field isomorphic to* $K(T)$ *and that the only proper specialization of* $\mathfrak{P}$ *in* $\mathrm{Sper} A$ *is* $\big\{f \in A \;\big|\; f(0) \ge 0\big\}$*. Also, show that the valuation group and residue field of* $\mathcal{R}_{A,\mathfrak{P}}$ *are* $\mathbb{Q}$ *and* $\mathfrak{l}$*, respectively!*

This establishes the claim made in Example 2.3.3 from the lecture.

Two of the 22 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted in the lecture Friday, July 5.