Tenth exercise sheet Advanced Algebra II.

Problem 1 (6 points). *Let K be a real closed field which is Archimedean and $I \subseteq K[X_1, \ldots, X_n]$ an ideal such that $X = \mathrm{Sper}K[X_1, \ldots, X_n]/I$ is K-proper. Construct a bijection $X^{\mathrm{Ber}} \xrightarrow{\cong} V_{\mathbb{R}^n}(I)$!*

For success of the next exercise this ought to be done in such a way that the bijection is functorial in $A = K[X_1, \ldots, X_n]/I$, although other solutions will still get points.

Problem 2 (4 points). *Show that the bijection from your solution to Problem 1 is a homeomorphism, where $V_{\mathbb{R}^n}(I)$ is equipped with the topology induced from $\mathbb{R}^n$!*

Problem 3 (2–4 points). *In the situation of problem 7 from sheet 9, show that $\mathfrak{P}_{\trianglelefteq}^{(\infty)}$ is a closed point of* $\mathrm{Sper}A$*!*

Remark 1. *The problem can be solved directly, by showing that there is no larger cone which is still proper, or by using a more recent criterion for closed points from the lecture. Three points will be awarded for giving two correct and independent proofs, two points for a single correct proof.*

In the situation of problem 6 from sheet 9 it is easy to see that there is a unique structure of an ordered group on $\mathbb{Q}^m$ extending the ordering $\trianglelefteq$ of $\mathbb{N}^m$. This ordering will also be denoted by $\trianglelefteq$. It is also easy to see that $\mathrm{supp}\mathfrak{P}_{\trianglelefteq}^{(0)} = \{0\}$. Let R be the convex hull of A in its field of quotients K and $\mathcal{R}$ its convex hull in $\mathfrak{K}(\mathfrak{P}_{\trianglelefteq}^{(0)})$.

Problem 4 (2 points). *In the situation of problem 6 from sheet 9, identify the valuation group of R with $(\mathbb{Z}^m, \trianglelefteq)$!*

Problem 5 (2 points). *In the situation of problem 6 from sheet 9, identify the valuation group of $\mathcal{R}$ with $(\mathbb{Q}^m, \trianglelefteq)$!*

In the situation of the previous two exercises, consider the maps

(1)
$$\mathrm{Spec}\mathcal{R} \cong \mathrm{Spec}R \xrightarrow{(2.3.1)} \left\{\mathfrak{q} \in \mathrm{Spec}A \;\middle|\; \mathfrak{q} \text{ is } \mathfrak{P}_{\trianglelefteq}^{(0)}\text{-convex}\right\} \cong \overline{\{\mathfrak{P}_{\trianglelefteq}^{(0)}\}}$$

where the leftmost map is induced by $R \to \mathcal{R}$ and is a bijection by problem 5 from sheet 8. The middle map sends $\mathfrak{r} \in \mathrm{Spec}\mathcal{R}$ to its preimage in A and is surjective by Proposition 2.3.4. from the lecture. The rightmost map sends $\mathfrak{Q}$ to $\mathrm{supp}\mathfrak{Q}$ and is a bijection by Proposition 2.3.2 from the lecture. Every convex subgroup $\Gamma \subseteq \mathbb{Z}^m$ defines a prime ideal $\mathfrak{r}_\Gamma$ of R in view of the previous two problems and problem 1 from sheet 8.

Problem 6 (2 points)**.** *Show that the preimage of* $\mathfrak{r}_\Gamma$ *in* A *depends only on* $\Gamma \cap \mathbb{N}^m$*!*

Problem 7 (2 points)**.** *Let* $\trianglelefteq$ *be the lexicographic orderd:* $(a, b) \trianglelefteq (\alpha, \beta)$ *if and only if* $a < \alpha$ *or* $a = \alpha$ *and* $b \leq \beta$*. Show that (1) is a bijection in this case.*

Problem 8 (2 points)**.** *Let* $(a, b) \trianglelefteq (\alpha, \beta)$ *if and only if* $a + b < \alpha + \beta$ *or* $a + b = \alpha + \beta$ *and* $a < \alpha$*. Show that in this case the left hand side of (1) has three and the right hand side two elements!*

This establishes the claim made in Example 2.3.2 from the lecture.

Four of the up to 24 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted in the lecture Friday, June 28.