Algebraic Dynamical Systems

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Multiparameter Ergodic Theory

Classical ergodic theory typically deals with single transformations and one-parameter flows (i.e. actions of $\mathbb{Z}, \mathbb{N}, \mathbb{R}$ or $\mathbb{R}^+$) on measure spaces which are usually interpreted as time evolutions.

Multiparameter ergodic theory is rooted in statistical mechanics (lattice models), mathematical biology (cellular automata) and probability theory (percolation models) and studies spatially extended systems with multidimensional symmetry groups (e.g. $\mathbb{Z}^{\mathbf{d}}$ or $\mathbb{R}^{\mathbf{d}}$ with $d > 1$). Explicit examples of such actions are often very difficult to analyze, and general questions about them (e.g. about multi-dimensional *shifts of finite type*) lead to undecidability problems which effectively prevent progress in a general setting.

Furstenberg (1967) considered the $\mathbb{N}^2$-action on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ generated by multiplication by 2 and by 3. He proved that $\mathbb{T}$ is the only infinite closed invariant set for this action and asked whether Lebesgue measure is the only nonatomic probability measure on $\mathbb{T}$ which is invariant under this action.

This question is still unanswered, but it led to to considerable interest in $\mathbb{Z}^{\mathbf{d}}$- and $\mathbb{R}^{\mathbf{d}}$-actions arising from arithmetical or algebraic settings.

Some Properties of $\mathbb{Z}^d$-Actions on Tori and Solenoids

Furstenberg's paper soon led to the realization that (*irreducible* and *mixing*) $\mathbb{Z}^{\mathbf{d}}$-actions by automorphisms of tori or solenoids have unusual properties.

- Invariant sets: The anaologue of Furstenberg's result about $\times 2$- and $\times 3$-invariant sets holds for such actions (Berend, 1983–1984).
- Invariant measures: Lebesgue measure is the only invariant probability measure for such an action which has positive entropy under at least one element of the action (Rudolph, 1990; Katok-Spatzier, 1996–1998; Einsiedler-Lindenstrauss, 2003). This property is sometimes referred to as measure rigidity.
- Commutants and isomorphisms: Any measurable conjugacy between such actions is affine, and measurably conjugate actions are therefore algebraically conjugate (Katok-Katok-S, 2002). In particular, the centralizers of such actions are very small. This property is called isomorphism rigidity.
- Cohomology: The first cohomology of such actions is very restricted (e.g. Katok-Spatzier, 1997). This is a form of cohomological rigidity.

Are these 'rigidity properties' specific this class of $\mathbb{Z}^{\mathbf{d}}$-actions, or do they occur more widely? To answer this question one needs wider classes of examples.

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Sources of Algebraic Examples

- Choose a a free abelian group G of units in an algebraic number field k and let them act by multiplication on the additive group $\mathfrak{o}_{\mathbf{k}}$ of integers in k. The Pontryagin dual of $\mathfrak{o}_{\mathbf{k}}$ is a finite-dimensional torus $\mathbb{T}^{\mathbf{n}}$, and the automorphisms of $\mathbb{T}^{\mathbf{n}}$ dual to multiplication by the elements of G form a $\mathbb{Z}^{\mathbf{d}}$-action by toral automorphisms. This construction gives all irreducible $\mathbb{Z}^{\mathbf{d}}$-actions by toral automorphisms — at least up to finite-to-one factor maps.
- If G is replaced by a finitely generated free abelian multiplicative subgroup of $k^{\times}$ and $\mathfrak{o}_{\mathbf{k}}$ by the additive group $R_{\mathbf{S}}$ of S-units for a set S of places of k which contains all infinite places and for which $G \subset R_{\mathbf{S}}$, then one obtains all irreducible $\mathbb{Z}^{\mathbf{d}}$-actions by automorphisms of solenoids — again up to finite-to-one factor maps.
- Another class of examples is obtained by letting abelian subgroups of Lie groups act on homogeneous spaces. The study of such actions and their remarkable applications to diophantine problems will be the subject of Elon Lindenstrauss' lecture.

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Entropy Rank

One of the properties of all these $\mathbb{Z}^{\mathbf{d}}$-actions, as well as of $\mathbb{Z}^{\mathbf{d}}$-actions by commuting diffeomorphisms of manifolds, is that every element of the action has finite entropy, which forces the entropy of the $\mathbb{Z}^{\mathbf{d}}$-action (and of every $\mathbb{Z}^{\mathbf{r}}$-sub-action with $2 \leq r \leq d$) to be zero. In a certain sense these actions are 'small': their *entropy ranks* are ≤ 1.

Among d-dimensional shifts of finite type (for example) it is easy to find examples with positive entropy or with intermediate entropy rank $1 < r < d$, but these examples are usually very difficult to work with.

In the late 1980's a class of $\mathbb{Z}^{\mathbf{d}}$-actions emerged which *does* lend itself to systematic study, but which does not suffer from the entropy restrictions of smooth or arithmetical actions: algebraic $\mathbb{Z}^{\mathbf{d}}$-actions, i.e. $\mathbb{Z}^{\mathbf{d}}$-actions by automorphisms of compact abelian groups.

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Algebraic $\mathbb{Z}^d$-actions

If X is a compact abelian group with Pontryagin dual $\hat{X} = \mathrm{Hom}(X, \mathbb{S})$ (where $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$), then every continuous automorphism β of X induces a dual automorphism $\hat{\beta}$ of $\hat{X}$ and vice versa:

$$\langle a, \beta x\rangle = \langle \hat{\beta} a, x\rangle$$

for every $x \in X$ and $a \in \hat{X}$, where $\langle a, x\rangle$ stands for $a(x)$.

If $\alpha \colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ is an algebraic $\mathbb{Z}^{\mathbf{d}}$-action on a compact abelian group X, then the dual action $\hat{\alpha} \colon \mathbf{n} \mapsto \hat{\alpha}^{\mathbf{n}}$ turns $\hat{X}$ into a module over the group ring $R_{\mathbf{d}} = \mathbb{Z}[\mathbb{Z}^{\mathbf{d}}] \cong \mathbb{Z}[u_1^{\pm 1}, \dots, u_{\mathbf{d}}^{\pm 1}]$ with

$$\langle u^{\mathbf{n}} \cdot a, x\rangle = \langle \hat{\alpha}^{\mathbf{n}} a, x\rangle = \langle a, \alpha^{\mathbf{n}} x\rangle, \quad \mathbf{n} = (n_1, \dots, n_{\mathbf{d}}) \in \mathbb{Z}^{\mathbf{d}},\ a \in \hat{X},\ x \in X,$$

where $u^{\mathbf{n}} = u_1^{\mathbf{n}_1} \cdots u_{\mathbf{d}}^{\mathbf{n}_d}$. This is the dual module of α.

Conversely, if M is a module over $R_{\mathbf{d}}$, the dual group $X_{\mathbf{M}} = \hat{M}$ carries the algebraic $\mathbb{Z}^{\mathbf{d}}$-action $\alpha_{\mathbf{M}}$ with

$$\langle u^{\mathbf{n}} \cdot a, x\rangle = \langle a, \alpha_{\mathbf{M}}^{\mathbf{n}} x\rangle$$

for every $\mathbf{n} \in \mathbb{Z}^{\mathbf{d}}$, $x \in X_{\mathbf{M}}$ and $a \in M$. In other words, algebraic $\mathbb{Z}^{\mathbf{d}}$-actions are in one-to-one correspondence with modules over the group ring $R_{\mathbf{d}}$ of $\mathbb{Z}^{\mathbf{d}}$.

Examples and Properties of Algebraic $\mathbb{Z}^d$-actions

Example. The simplest $R_{\mathbf{d}}$-modules are those of the form $R_{\mathbf{d}}/I$, where $I \subset R_{\mathbf{d}}$ is an ideal. We denote by σ the shift-action

$$(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$$

of $\mathbb{Z}^{\mathbf{d}}$ on $\mathbb{T}^{\mathbb{Z}^d}$ and set $f(\sigma) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}}\sigma^{\mathbf{n}}$ for every $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$. Then

$$X_{\mathbf{R}_d/\mathbf{I}} = \{x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma) = 0 \text{ for all } f \in I\} = \bigcap_{\mathbf{f}\in\mathbf{I}} \ker f(\sigma),$$

and $\alpha_{\mathbf{R}_d/\mathbf{I}}$ is the restriction of σ to $X_{\mathbf{R}_d/\mathbf{I}}$.

- For $I = (2) = 2R_{\mathbf{d}}$, $X_{\mathbf{R}_d/\mathbf{I}} = \{x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : 2x_{\mathbf{n}} = 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^{\mathbf{d}}\}$.
- For $d = 2$, $f = 1 + u_1 + u_2$ and $I = (f) = fR_2$,
  $X_{\mathbf{R}_2/(\mathbf{f})} = \{x \in \mathbb{T}^{\mathbb{Z}^2} : x_{(\mathbf{n}_1,\mathbf{n}_2)} + x_{(\mathbf{n}_1+1,\mathbf{n}_2)} + x_{(\mathbf{n}_1,\mathbf{n}_2+1)} = 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^2\}$.

The correspondence between modules und algebraic $\mathbb{Z}^{\mathbf{d}}$-actions yields a ‘dictionary’ between algebraic properties of the module M and dynamical properties of the algebraic $\mathbb{Z}^{\mathbf{d}}$-action $\alpha_{\mathbf{M}}$. In this lecture I'll restrict myself to a just a few entries in this dictionary: entropy, expansiveness, mixing and some aspects of the isomorphism problem.

Entropy and Expansiveness

Let X be a compact set and $T\colon \mathbf{n} \mapsto T^{\mathbf{n}}$ a continuous $\mathbb{Z}^{\mathsf{d}}$-action on X. If $\mathcal{U}$ is an open cover of X we set $N(\mathcal{U}) = \min_{\mathcal{C}} |\mathcal{C}|$, where $\mathcal{C}$ ranges over the finite subcovers of $\mathcal{U}$. The topological entropy of T is defined as

$$h(T) = \sup_{\mathcal{U}} \lim_{\mathsf{n}\to\infty} \frac{1}{|Q_{\mathsf{n}}|} \log N\Bigl(\bigvee_{\mathbf{n}\in \mathsf{Q}_n} T^{-\mathbf{n}}(\mathcal{U})\Bigr),$$

where $\mathcal{U}$ ranges over the (finite) open covers of X, $\mathcal{U}_1 \vee \mathcal{U}_2 = \{U \cap V : U \in \mathcal{U}_1,\, V \in \mathcal{U}_2\}$ and $(Q_{\mathsf{n}},\, n \geq 1)$ is any sequence of *rectangles* in $\mathbb{Z}^{\mathsf{d}}$ increasing to $\mathbb{Z}^{\mathsf{d}}$.

The action T is expansive if there exists an open neighbourhood V of the diagonal $\Delta \subset X \times X$ such that

$$\bigcap_{\mathbf{n}\in\mathbb{Z}^d} (T^{-\mathbf{n}} \times T^{-\mathbf{n}})(V) = \Delta.$$

If δ is a metric on X then expansiveness is equivalent to the existence of an expansive constant, i.e. an $\varepsilon > 0$ such that any two points $x, y \in X$ with $\delta(T^{\mathbf{n}}x, T^{\mathbf{n}}y) < \varepsilon$ for all $\mathbf{n} \in \mathbb{Z}^{\mathsf{d}}$ coincide.

If T is expansive there exists a finite open cover $\mathcal{U}$ of X with

$$h(T) = \lim_{\mathsf{n}\to\infty} \frac{1}{|Q_{\mathsf{n}}|} \log N\Bigl(\bigvee_{\mathbf{n}\in \mathsf{Q}_n} T^{-\mathbf{n}}(\mathcal{U})\Bigr).$$

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Entropy and Expansiveness of Algebraic Actions

Let $I \subset R_{\mathbf{d}}$ be an ideal, and let

$$V_{\mathbb{C}}(I) = \{\mathbf{c} \in (\mathbb{C}^{\times})^{\mathbf{d}} : f(\mathbf{c}) = 0 \text{ for every } f \in I\}$$

be the variety of I.

The $\mathbb{Z}^{\mathbf{d}}$-action $\alpha_{\mathbf{R}_d/\mathbf{I}}$ is expansive if and only if $V_{\mathbb{C}}(I) \cap \mathbb{S}^{\mathbf{d}} = \varnothing$. More generally, if α is an algebraic $\mathbb{Z}^{\mathbf{d}}$-action, then α is expansive if and only if its dual module M is Noetherian and $\alpha_{\mathbf{R}_d/\mathfrak{p}}$ is expansive for every *associated prime ideal* of M (S, 1990).

The entropy of $\alpha_{\mathbf{R}_d/\mathbf{I}}$ is given by

$$h(\alpha_{\mathbf{R}_d/\mathbf{I}}) = \begin{cases} 0 & \text{if } I \text{ is nonprincipal,} \\ \mathsf{m}(f) = \int_{\mathbb{S}^d} \log |f(\mathbf{s})| \, d\lambda_{\mathbb{S}^d}(\mathbf{s}) & \text{if } I = (f) = f \cdot R_{\mathbf{d}}, \end{cases}$$

(Lind-S-Ward, 1990).

Example (Kronecker, 1857). If $f \in R_{\mathbf{d}}$ then $\mathsf{m}(f) = 0$ if and only if f is a product of generalized cyclotomic polynomials (i.e. of polynomials of the form $u^{\mathbf{m}} c(u^{\mathbf{n}})$, where c is cyclotomic). In this case $\alpha_{\mathbf{R}_d/(\mathbf{f})}$ is nonexpansive.

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Examples

Example (Smyth, 1981). Let $d=3$ and $f=1+u_1+u_2+u_3$. Then $\alpha_{\mathbf{R}_3/(\mathbf{f})}$ is nonexpansive,

$$X_{\mathbf{R}_3/(\mathbf{f})} = \{x=(x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^3} : x_{(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)} + x_{(\mathbf{n}_1+1,\mathbf{n}_2,\mathbf{n}_3)} + x_{(\mathbf{n}_1,\mathbf{n}_2+1,\mathbf{n}_3)}$$
$$+ x_{(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3+1)} = 0 \text{ for all } (n_1,n_2,n_3) \in \mathbb{Z}^3\},$$

and

$$h(\alpha_{\mathbf{R}_3/(\mathbf{f})}) = \frac{7}{2\pi^2}\zeta(3),$$

where $\zeta(3) = \sum_{\mathbf{n}=1}^{\infty} n^{-3}$.

Example (Smyth, 1981). Let $d=2$ and $f=1+u_1+u_2$. Then $\alpha_{\mathbf{R}_2/(\mathbf{f})}$ is nonexpansive and

$$h(\alpha_{\mathbf{R}_2/(\mathbf{f})}) = \int_0^1\int_0^1 \log|1+e^{2\pi \mathbf{i s}}+e^{2\pi \mathbf{i t}}|\,ds\,dt = \frac{3\sqrt{3}}{4\pi}L(2,\chi_3),$$

where

$$\chi_3(m) = \begin{cases} 0 & \text{if } m\equiv 0 \pmod 3, \\ 1 & \text{if } m \equiv 1 \pmod 3, \\ -1 & \text{if } m\equiv 2 \pmod 3, \end{cases} \quad \text{and} \quad L(s,\chi) = \sum_{\mathbf{n}=1}^{\infty}\frac{\chi(n)}{n^{\mathbf{s}}} = \prod_{\mathbf{p}\ \text{prime}}\left(1-\frac{\chi(p)}{p^{\mathbf{s}}}\right)^{-1}.$$

Isomorphisms

We turn to the measurable conjugacy problem for algebraic $\mathbb{Z}^{\mathbf{d}}$-actions, i.e. the question when two algebraic $\mathbb{Z}^{\mathbf{d}}$-actions α and β on compact abelian groups X and Y are measurably conjugate: when does there exist a measure-preserving almost one-to-one Borel map $\phi\colon X \longrightarrow Y$ with

$$\phi \circ \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \circ \phi \;\; \lambda_{\mathbf{X}}\text{-a.e. for every } \mathbf{n} \in \mathbb{Z}^{\mathbf{d}}?$$

The conjugacy ϕ in this equation is algebraic if it is a.e. equal to an affine map.

If α and β are measurably conjugate, then their entropies coincide. If α and β are both *Bernoulli*, then coincidence of entropy is also sufficient for conjugacy (Ornstein-Weiss, 1987). This raises the question how one can recognize algebraic $\mathbb{Z}^{\mathbf{d}}$-actions which are Bernoulli.

Theorem (Lind-S-Ward, 1990; Rudolph-S, 1995). An algebraic $\mathbb{Z}^{\mathbf{d}}$-action α with dual module M is Bernoulli if and only if $h(\alpha_{\mathbf{R}_d/\mathfrak{p}}) > 0$ for every prime ideal $\mathfrak{p}$ associated with M.

It is not known whether there always exist finitary (or 'nice') isomorphism between Bernoulli actions of equal entropy.

Where the isomorphism problem becomes really interesting is for actions with zero entropy.

Two Examples

Example. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \in \mathsf{GL}(3, \mathbb{Z})$$

has the irreducible characteristic polynomial

$$f = x^3 - 3x - 1$$

with roots

$$a_1 = -1.532\dots, \; a_2 = -0.3473\dots, \; a_3 = 1.879\dots$$

Hence A has a two-dimensional expanding subspace, but the expanding subspace of A^{-1} has dimension 1. It follows that A and A^{-1} are not topologically conjugate, although they are measurably conjugate.

Example. Let α be the $\mathbb{Z}^2$-action on $\mathbb{T}^3$ given by $\alpha^{\mathbf{n}} = A^{\mathbf{n}_1} B^{\mathbf{n}_2}$ for $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, where

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -4 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 2 \\ 2 & 5 & -2 \end{pmatrix}.$$

Let

$$V = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -3 & 0 \\ 1 & -4 & -2 \end{pmatrix},$$

$$A' = V^{-1} A V = \begin{pmatrix} 2 & -4 & -1 \\ 1 & -4 & -1 \\ 1 & -5 & -1 \end{pmatrix}, \qquad B' = V^{-1} B V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 6 & -3 \end{pmatrix},$$

and let α' be the $\mathbb{Z}^2$-action $\mathbf{n} \mapsto A'^{\mathbf{n}_1} B'^{\mathbf{n}_2}$. Are α and α' measurably conjugate?

Isomorphism Rigidity

Theorem. Let α and β be *irreducible* and *mixing* algebraic $\mathbb{Z}^{\mathbf{d}}$-actions with $d > 1$ on compact connected abelian groups X and Y. Then α and β are measurably conjugate if and only if they are algebraically conjugate, and any measurable conjugacy is affine.

The actions α and α' above are therefore not measurably conjugate.

Here is the most general version of this result.

Theorem. Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^{\mathbf{d}}$-actions with zero entropy. Suppose that there exists an infinite subgroup $\Gamma \subset \mathbb{Z}^{\mathbf{d}}$ such that the restrictions of α and β to Γ are Bernoulli with finite entropy. Then every measurable conjugacy between α and β is affine.

The hypothesis concerning finite entropy Bernoulli subactions is necessary: if it is not satisfied there can exist non-affine isomorphisms, and measurably conjugate actions need not be algebraically conjugate (Bhattacharya, 2003). For algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on zero-dimensional groups the situation is well understood, but for actions on connected groups this is still an open problem.

Remarks about the Isomorphism Rigidity Theorem

Every measurable conjugacy of algebraic $\mathbb{Z}^{\mathbf{d}}$-actions α and β on compact abelian groups X and Y defines an $(\alpha \times \beta)$-invariant probability measure μ on $X \times Y$ which projects onto $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$ under the coordinate projections (such a measure is a joining of $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$).

The isomorphism rigidity theorem for mixing $\mathbb{Z}^{\mathbf{d}}$-actions by toral and solenoidal automorphisms is a consequence of the scarcity of $(\alpha \times \beta)$-invariant probability measures which have positive entropy under some element of the action (which is guaranteed by the hypothesis that μ is a joining of $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$).

For algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on zero-dimensional groups the result is due to Kitchens-S (2000) in the irreducible case and to Bhattacharya-S (2003) in the general case. For this class of actions isomorphism rigidity is closely linked to the breakdown of higher order mixing for such actions.

Mixing

A measure-preserving $\mathbb{Z}^{\mathbf{d}}$-action T on a probability space $(X, \mathcal{S}, \mu)$ is mixing of order $r \geq 2$ if

$$\lim_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_r \in \mathbb{Z}^d \\ \|\mathbf{n}_i - \mathbf{n}_j\| \to \infty \text{ for } 1 \leq \mathbf{i} < \mathbf{j} \leq \mathbf{d}}} \mu\Bigl(\bigcap_{\mathbf{i}=1}^{\mathbf{r}} T^{-\mathbf{n}_i} B_{\mathbf{i}}\Bigr) = \prod_{\mathbf{i}=1}^{\mathbf{r}} \mu(B_{\mathbf{i}})$$

for all Borel sets $B_{\mathbf{i}} \subset X$, $i = 1, \dots, r$.

The order of mixing of T is the largest value r such that T is r-mixing.

A nonempty finite subset $F \subset \mathbb{Z}^{\mathbf{d}}$ is mixing under T if

$$\lim_{\mathbf{k} \to \infty} \mu\Bigl(\bigcap_{\mathbf{n} \in \mathbf{F}} T^{-\mathbf{kn}} B_{\mathbf{n}}\Bigr) = \prod_{\mathbf{n} \in \mathbf{F}} \mu(B_{\mathbf{n}}) \tag{1}$$

for all Borel sets $B_{\mathbf{n}} \subset X$, $\mathbf{n} \in F$, and nonmixing otherwise.

Mixing on Connected Groups

Theorem (S-Ward, 1993). An algebraic $\mathbb{Z}^{\mathbf{d}}$-action α on a compact connected abelian group is 2-mixing if and only if it is r-mixing for every $r \geq 2$.

This statement turns out to be equivalent to a result about additive relations in fields (cf. v.d.Poorten-Schlickewei, 1991; Evertse-Schlickewei-Schmidt, 2002):

Theorem. Let K be a field of characteristic 0 and G a finitely generated multiplicative subgroup of $K^{\times} = K \smallsetminus \{0\}$. If $r \geq 2$ and $(c_1, \dots, c_{\mathbf{r}}) \in (K^{\times})^{\mathbf{r}}$, then the equation

$$\sum_{\mathbf{i}=1}^{\mathbf{r}} c_{\mathbf{i}} x_{\mathbf{i}} = 0$$

has only finitely many solutions $(x_1, \dots, x_{\mathbf{r}}) \in G^{\mathbf{r}}$ such that no sub-sum of this sum vanishes.

For algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on compact zero-dimensional abelian groups the mixing behaviour is quite different:

Theorem (S-Ward, 1993). Let α be an algebraic $\mathbb{Z}^{\mathbf{d}}$-action on a compact zero-dimensional abelian group. The following conditions are equivalent.

- α is r-mixing for every $r \geq 2$;
- Every finite set $F \subset \mathbb{Z}^{\mathbf{d}}$ is mixing under α;
- α is Bernoulli (and therefore has positive entropy).

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A Digression: Nonmixing Sets and the Order of Mixing

If α is an algebraic $\mathbb{Z}^{\mathbf{d}}$-action which has a nonmixing set $F \subset \mathbb{Z}^{\mathbf{d}}$, then α is obviously not $|F|$-mixing. Is the converse true?

Theorem (Masser, 2004). The order of mixing of an algebraic $\mathbb{Z}^{\mathbf{d}}$-action α is equal to the smallest cardinality of all nonmixing sets of α.

Although the collection of all nonmixing sets of such actions is quite complicated, this theorem, together with an earlier result by Masser on equations of the form

$$c_1 a_1^{\mathbf{k}} + \cdots + c_{\mathbf{r}} a_{\mathbf{r}}^{\mathbf{k}} = 1 \text{ for infinitely many } k \geq 0$$

over fields of positive characteristic, allows (at least in principle) to determine the precise order of mixing for algebraic $\mathbb{Z}^{\mathbf{d}}$-actions.

For nonalgebraic actions the problem of determining the order of mixing looks hopeless: even for a single ergodic transformation it is not known whether 2-mixing has to imply 3-mixing.

Proof of Isomorphism Rigidity

The following result depends on the existence of nonmixing sets and is crucial for isomorphism rigidity in the zero-dimensional setting.

Theorem (Bhattacharya, 2003) Every measurable equivariant map between zero entropy algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on compact zero-dimensional abelian groups is a.e. equal to a continuous map.

This result also provides detailed information about measurable conjugacies of actions without finite entropy Bernoulli subactions.

In 2005, Einsiedler provided a conceptually more direct proof of the isomorphism rigidity result for zero-dimensional groups using measure rigidity. The disadvantage of this approach is that it provides no information if there are no finite entropy Bernoulli subactions.

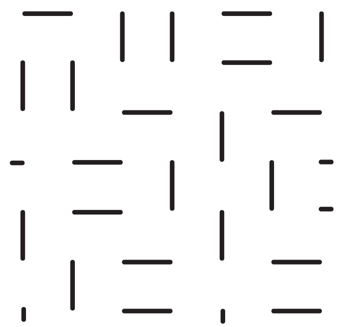
The final step in proving the isomorphism rigidity theorem is again due to Bhattacharya (Preprint, 2005) and uses measure rigidity of algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on finite-dimensional tori and solenoids to obtain isomorphism rigidity for mixing algebraic $\mathbb{Z}^{\mathbf{d}}$-actions on compact connected abelian groups with finite entropy Bernoulli subactions.

Connections Between Algebraic and Symbolic Systems

Example (cf. Kasteleyn, 1961; Burton-Pemantle, 1993, R. Solomyak, 1998). Let $d = 2$ and $f = 4 - u_1 - u_1^{-1} - u_1 - u_2^{-1}$ (this action will be called the harmonic model). Then $\alpha_{\mathbf{R}_2/(\mathbf{f})}$ is nonexpansive and

$$h(\alpha_{\mathbf{R}_2/(\mathbf{f})}) = \int_0^1 \int_0^1 \log(4 - 2\cos 2\pi s - 2\cos 2\pi t)\, ds\, dt = 4 \cdot h(\sigma_{\mathbf{D}}),$$

where $\sigma_{\mathbf{D}}$ is the shift-action of $\mathbb{Z}^2$ on the space of dimers consisting of all infinite configurations of exact pairings of elements in $\mathbb{Z}^2$ of the form



The dimer model is Bernoulli with respect to its unique *measure of maximal entropy*. Hence the harmonic model is measurably conjugate to the ‘even’ shift-action of $\mathbb{Z}^2$ on the space of dimers.

There is another lattice model with the same entropy as the harmonic model, the abelian sandpile model.

Construction of Conjugacies

These examples make it is desirable to find a method for constructing equivariant maps between symbolic and algebraic $\mathbb{Z}^{\mathbf{d}}$-actions.

The constructions of Markov partitions for toral automorphisms (Adler-Weiss, 1967; Bowen, 1970) amount to finding a (1-dimensional) SFT Y and a continuous, surjective and *almost injective* equivariant map from Y to the torus.

Following an idea of Vershik (Vershik, 1992; Sidorov-Vershik, 1998; cf. also Kenyon-Vershik, 1998), one can use *homoclinic points* to construct such covering maps of expansive toral automorphism algebraically (Einsiedler-S, 1997; Lind-S, 1999; S, 2000). This construction can be extended to expansive algebraic $\mathbb{Z}^{\mathbf{d}}$-actions of the form $\alpha_{\mathbf{R}_d/(\mathbf{f})}$, $f \in R_{\mathbf{d}}$.

Some differences between single automorphisms and $\mathbb{Z}^{\mathbf{d}}$-actions:

- For $d = 1$ the symbolic representation provides information about the group automorphism, but for $d > 1$ the algebraic $\mathbb{Z}^{\mathbf{d}}$-action is often understood better than the symbolic system covering it.
- There are no good symbolic covers for (irreducible) nonhyperbolic toral automorphisms (cf. e.g. Lindenstrauss-S, 2005), but such covers can exist for nonexpansive $\mathbb{Z}^{\mathbf{d}}$-actions (like the harmonic model).

Construction of Symbolic Covers

Let $\ell^\infty(\mathbb{Z}^{\mathbf{d}})$ be the space of bounded real-valued maps on $\mathbb{Z}^{\mathbf{d}}$, furnished with the shift action σ, and let $f(\sigma)\colon \ell^\infty(\mathbb{Z}^{\mathbf{d}}) \longrightarrow \ell^\infty(\mathbb{Z}^{\mathbf{d}})$ be given by

$$f(\sigma)(v) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}}\sigma^{\mathbf{n}}v, \quad v = (v_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^{\mathbf{d}}).$$

The map $f(\sigma)$ is injective if and only if $\alpha_{\mathbf{R}_d/(\mathbf{f})}$ is expansive.

If $\alpha = \alpha_{\mathbf{R}_d/(\mathbf{f})}$ is expansive there exists a fundamental homoclinic point $x^\Delta \in X_{\mathbf{R}_d/(\mathbf{f})}$ such that the map $\xi\colon \ell^\infty(\mathbb{Z}^{\mathbf{d}},\mathbb{Z}) \longrightarrow X = X_{\mathbf{R}_d/(\mathbf{f})}$, defined by

$$\xi(v) = \sum_{\mathbf{n}\in\mathbb{Z}^d} v_{\mathbf{n}} \cdot \alpha^{-\mathbf{n}}x^\Delta, \quad v = (v_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^{\mathbf{d}}),$$

is an equivariant surjective group homomorphism with kernel $f(\sigma)(\ell^\infty(\mathbb{Z}^{\mathbf{d}},\mathbb{Z}))$.

A Baire category argument shows that $\xi(V_{\mathbf{N}}) = X$ for some $N \geq 1$, where $V_{\mathbf{N}} = \{-N,\dots,N\}^{\mathbb{Z}^d} \subset \ell^\infty(\mathbb{Z}^{\mathbf{d}},\mathbb{Z})$.

By restricting ξ to $V_{\mathbf{N}}$ one sees that α satisfies a strong form of specification: given $\varepsilon > 0$ there exists $M > 0$ such that we can find, for any two regions $Q_1, Q_2 \subset \mathbb{Z}^{\mathbf{d}}$ with (Euclidean) distance $> M$, and for any $x^{(1)}, x^{(2)} \in X$, a $y \in X$ with $\delta(\alpha^{\mathbf{n}}y, \alpha^{\mathbf{n}}x^{(\mathbf{i})}) < \varepsilon$ for all $\mathbf{n} \in Q_{\mathbf{i}}$.

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Symbolic Covers and Entropy

This homomorphism ξ can be used to prove the entropy formula for expansive algebraic $\mathbb{Z}^{\mathbf{d}}$-actions described earlier and to extend (an appropriate version of) this formula to algebraic actions of residually finite amenable groups (Deninger, 2006; Deninger-S, 2006):

If $X^{(\mathbf{K})} = \{x \in X : \alpha^{\mathbf{kn}} x = x \text{ for all } \mathbf{n} \in \mathbb{Z}^{\mathbf{d}}\}$ is the set of points with period K in each coordinate, then expansiveness and specification imply that

$$h(\alpha) = \lim_{\mathbf{K} \to \infty} \frac{1}{K^{\mathbf{d}}} \log \left| X^{(\mathbf{K})} \right|.$$

Furthermore, if

$$V^{(\mathbf{K})} = \{v \in \ell^{\infty}(\mathbb{Z}^{\mathbf{d}}, \mathbb{Z}) : \sigma^{\mathbf{kn}} v = v \text{ for all } \mathbf{n} \in \mathbb{Z}^{\mathbf{d}}\},$$

then

$$X^{(\mathbf{K})} \cong V^{(\mathbf{K})} / f(\sigma)(V^{(\mathbf{K})}),$$

and $\left| X^{(\mathbf{K})} \right|$ is therefore the absolute value of a certain determinant, which turns out to be a Riemann approximation of the integral in the entropy formula.

Symbolic Representations

The problem one has to solve for symbolic representations of $X_{\mathbf{R}_d/(\mathbf{f})}$ is to find 'good' closed, shift-invariant subsets $V' \subset V_{\mathbf{N}}$ which intersect each coset of $f(\sigma)(\ell^\infty(\mathbb{Z}^{\mathbf{d}}, \mathbb{Z})) \subset \ell^\infty(\mathbb{Z}^{\mathbf{d}}, \mathbb{Z})$ in at least one point, and such that the restriction of ξ to V' is almost injective. In particular, the shift-action of $\mathbb{Z}^{\mathbf{d}}$ on V' must have the same entropy as $\alpha_{\mathbf{R}_d/(\mathbf{f})}$.

The first example of this kind by Vershik yielded a Markov partition for the hyperbolic toral automorphism $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Here is a simple two-dimensional **Example** (Einsiedler-S, 1997).
Let $d = 2$ and $f = 3 - u_1 - u_2$. Then the restriction of $\xi\colon \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) \longrightarrow X_{\mathbf{R}_2/(\mathbf{f})}$ to $\{0, 1, 2\}^{\mathbb{Z}^2} \subset \ell^\infty(\mathbb{Z}^2, \mathbb{Z})$ is surjective and *almost injective*.

This implies the existence of an *almost topological conjugacy* between $\alpha_{\mathbf{R}_2/(\mathbf{f})}$ and, for example, $\alpha_{\mathbf{R}_2/(\mathbf{f}')}$ with $f' = 3 - u_1^{-1} - u_2^{-1}$.

There are further examples and some very interesting problems of this kind. The main technical difficulties in proving results in this area are that, for $d > 1$, the natural candidates for equal entropy covers may be quite complicated (they need not be SFT's), and that the covering maps need not be finite-to-one.