Stochastic Partial Differential Equations and Infinite Dimensional Analysis

Michael Röckner

joint work (several papers) with:

Viorel Barbu, Vladimir I. Bogachev,
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Ref.: BiBoS-Preprint Server, my homepage at Purdue University
(BiBoS = Bielefeld–Bonn–Stochastics Research Centre)

A – From ODE to PDE

in finitely many variables

A – ODE to PDE, finite dimensional 1a

ODE

$$(1)\qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad\qquad \text{on } \mathbb{R}^d$$

$$X_t^x \quad = x + \int_0^t B\big(X_s^x \quad\big)\,\mathrm{d}s$$

$$\Rightarrow \qquad \begin{aligned} &p_t f(x) := \quad f\big(X_t^x \quad\big) \\ &p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by flow property

solves **PDE** (2)

$$\Rightarrow \qquad \begin{aligned} p_t f(x) &:= \quad f\big(X_t^x \quad\big) \\ p_0 f &= f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by flow property

solves **PDE**

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) = \sum_{i=1}^{d} B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x)$$

$$(1) \qquad = B(x) \cdot \nabla_x p_t f(x)$$

$$=: \underset{\substack{\uparrow \\ \text{"generator"} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \ldots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \ldots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \ldots, 0)$ canonical basis

ODE

$$(1) \qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d$$

$$X_t^x \quad = x + \int_0^t B\big(X_s^x \quad\big)\,\mathrm{d}s$$

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semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by flow property

solves **PDE** (2)

A – SODE to PDE, finite dimensional

SODE

$$
(1) \qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t + \mathrm{d}W_t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d
$$

$$
X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \underbrace{W_t(\omega)}_{\substack{\text{Brownian motion}\\ \text{on } \mathbb{R}^d}}
$$

$$
\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad \begin{aligned} &p_t f(x) := \int f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big] \\ &p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R} \end{aligned}
$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (2) (heat equation in finitely many variables)

$$\Rightarrow \qquad p_t f(x) := \quad f\big(X_t^x \quad\big)$$
$$p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R}$$

semigroup
$\big(p_{t+s} f(x) = p_t(p_s f)(x)\big)$
by flow property

solves **PDE**

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) = \sum_{i=1}^{d} B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x)$$

$$(2) \qquad = B(x) \cdot \nabla_x p_t f(x)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \dots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \dots, 0)$ canonical basis

A – SODE to PDE, finite dimensional

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad p_t f(x) := \int f\big(X_t^x(\omega)\big)\, \mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big]$$

$$p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R}$$

semigroup
$\big(p_{t+s} f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (heat equation in finitely many variables)

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) \overset{\text{Itô}}{=} \sum_{i=1}^{d} B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x) + \frac{1}{2} \sum_{i\ =1}^{d} \qquad \frac{\partial^2}{\partial e_i \partial e_i} p_t f(x)$$

$$(3) \qquad = B(x) \cdot \nabla_x p_t f(x) + \frac{1}{2} \Delta_x p_t f(x)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \dots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \dots, 0)$ canonical basis

Itô:

$$
\begin{aligned}
p_t f(x) &= E[f(X_t^x)] \\
&= f(x) + \sum_{i=1}^d \underbrace{E\left[\int_0^t \left(\frac{\partial}{\partial e_i} f\right)(X_s^x)\ dW_s^i\right]}_{=0} + \sum_{i=1}^d \int_0^t \underbrace{E\left[\left(\frac{\partial}{\partial e_i} f\right)(X_s^x)\ B^i(X_s^x)\right] ds}_{p_s\left(\left(\frac{\partial}{\partial e_i} f\right)\ B^i\right)(x) ds} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \underbrace{E\left[\left(\frac{\partial^2}{\partial e_i \partial e_j} f\right)(X_s^x)\ \underbrace{\langle dW_s^i, dW_s^j\rangle_{\mathbb{R}^d}}_{=\delta_{ij} ds}\right]}_{=p_s\left(\frac{\partial^2}{\partial e_i \partial e_j} f\right)(x)\ \delta_{ij} ds} \qquad \text{(Taylor up to order 2!)} \\
&= f(x) + \int_0^t \underbrace{p_s(Lf)(x)}_{L(p_s f)(x)}\ ds
\end{aligned}
$$

A – SODE to PDE, finite dimensional

SODE

$$(1)\qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t + \mathrm{d}W_t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d$$

$$X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \underbrace{W_t(\omega)}_{\substack{\text{Brownian motion} \\ \text{on } \mathbb{R}^d}}$$

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad \begin{aligned} &p_t f(x) := \int f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big] \\ &p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (2) (heat equation in finitely many variables)

A – SODE to PDE, finite dimensional **3a**

SODE

$$(1)\qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}W_t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d$$

$$X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \int_0^t \sigma\big(X_s^x(\omega)\big)\,\underbrace{\mathrm{d}W_{\cancel{t}\,s}(\omega)}_{\substack{\text{Brownian motion}\\ \text{on } \mathbb{R}^d}}$$

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad \begin{aligned} p_t f(x) &:= \int f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big] \\ p_0 f &= f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (2) (heat equation in finitely many variables)

A – SODE to PDE, finite dimensional 3b

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \quad p_t f(x) := \int f\big(X_t^x(\omega)\big)\, \mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big]$$

$$p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R}$$

semigroup
$\big(p_{t+s} f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (heat equation in finitely many variables)

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) \overset{\text{Itô}}{=} \sum_{i=1}^{d} B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x) + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial e_i \partial e_i} p_t f(x)$$

$$(4) \qquad = B(x) \cdot \nabla_x p_t f(x) + \frac{1}{2} \operatorname{Tr}\big(D^2 p_t f(x)\big)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \ldots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \ldots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \ldots, 0)$ canonical basis

A – SODE to PDE, finite dimensional 3b

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad p_t f(x) := \int f\big(X_t^x(\omega)\big)\, \mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big]$$

$$p_0 f = f \quad \text{for } f : \mathbb{R}^d \to \mathbb{R}$$

semigroup
$\big(p_{t+s} f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (heat equation in finitely many variables)

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) \overset{\text{Itô}}{=} \sum_{i=1}^{d} B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x) + \frac{1}{2} \sum_{i,j=1}^{d} \big(\sigma^T(x)\sigma(x)\big)^{ij} \cdot \frac{\partial^2}{\partial e_i \partial e_j} p_t f(x)$$

$$(5) \qquad = B(x) \cdot \nabla_x p_t f(x) + \frac{1}{2} \mathrm{Tr}\big(\sigma^T(x)\sigma(x) D^2 p_t f(x)\big)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \dots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \dots, 0)$ canonical basis

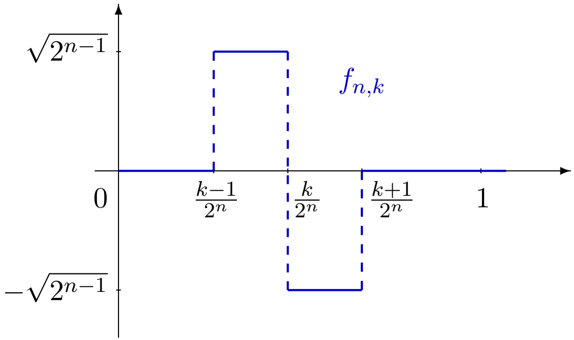
and $\sigma = (\sigma^{ij}) : \mathbb{R}^d \to \underbrace{M(d \times d)}_{d \times d\text{-matrices}}$

[Levy–Wiener–Ciesielski]

First ingredient: Haarbasis of $L^2\big([0,1],\,\mathrm{d}t\big)$:

Lebesgue measure

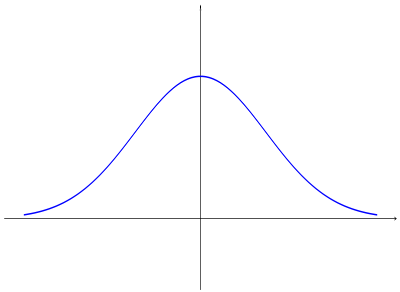
$f_{0,0} \equiv 1$, and for $n \in \mathbb{N}$, $0 < k < 2^n$, k odd,



$\big(f_{n,k}\big)_{\substack{0<k<2^n,\,k \text{ odd} \\ n\in\mathbb{N}}}$ is ONB of $L^2\big([0,1],\,\mathrm{d}t\big)$

Second ingredient: Standard normal distribution on $\mathbb{R}^\infty$

Standard normal distribution on $\mathbb{R}^1$:



$$\underset{\substack{\uparrow \\ \text{Gauss}}}{\gamma}(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \underset{\substack{\uparrow \\ \text{Lebesgue meas.} \\ \text{on } \mathbb{R}^1}}{\mathrm{d}x}$$

Set $\gamma_{n,k} := \gamma$.

$$\mathbb{P} := \bigotimes_{\substack{0<k<2^n \\ k \text{ odd} \\ n\in\mathbb{N}}} \gamma_{n,k} \qquad \text{product measure on } \mathbb{R}^\infty \ \big(= \mathbb{R}^{\{(n,k)|\ldots\}}\big)$$

Define $\xi_{n,k} : \mathbb{R}^{\{(n,k)|n\in\mathbb{N},\, 0<k<2^n,\, k \text{ odd}\}\cup\{(0,0)\}} \longrightarrow \mathbb{R}$ (projection) and for $t \in [0,1]$

$$\underset{\substack{\uparrow \\ \textbf{Brownian motion} \text{ on } \mathbb{R}^1}}{W_t(\omega)} := \sum_{(n,k)} \xi_{n,k}(\omega) \int_0^t f_{n,k}(s)\,\mathrm{d}s \qquad \text{converges for } \mathbb{P}\text{-a.e. } \omega \in \mathbb{R}^\infty$$

B – From SODE to PDE

in infinitely many variables

SODE in E

(1) $\quad \mathrm{d}X_t^x = B(X_t^x)\,\mathrm{d}t + \underbrace{\sigma(X_t^x)\,\mathrm{d}W_t}_{}$

Brownian motion on E

$X_0^x = x \in E \underset{\text{(e.g.)}}{:=}$ separable Hilbert space $\Big(\underset{\text{(e.g.)}}{=} L^2(\mathbb{R}^m, \mathrm{d}x), \quad H^k(\mathbb{R}^m, \mathrm{d}x), \quad \dots \Big)$

inner product $\langle\ ,\ \rangle$

Lebesgue measure

Sobolev space

under very strong conditions on B, σ

$\Longleftrightarrow$

OUR POINT!

$$p_t f(x) := \int_\Omega f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega)\,, \qquad p_0 f = f, \quad \text{for } f: E \to \mathbb{R}$$

solves **PDE** (2) (heat equation in infinitely many variables)

$\Longleftrightarrow$

$$p_t f(x) := \int_\Omega f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega)\,, \qquad p_0 f = f, \quad \text{for } f : E \to \mathbb{R}$$

solves **PDE** (heat equation in infinitely many variables)

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) = \sum_{i=1}^{\infty} \langle B(x)\,,\ e_i\rangle\,\frac{\partial}{\partial e_i} p_t f(x) + \frac{1}{2}\sum_{i,j=1}^{\infty}\langle \underbrace{\sigma^T(x)\sigma(x)}_{=:A(x)}\, e_i\,,\ e_j\rangle\,\frac{\partial^2}{\partial e_i \partial e_j} p_t f(x) \text{ (heuristically!)}$$

$$=: \underset{\substack{\uparrow\\ \text{“generator”}\\ \text{of (1)}}}{L}(p_t f)(x).$$

Here $B : E \to E$ and $\{e_i \mid i \in \mathbb{N}\}$ ONB of E and

$$\sigma : E \to \underset{\substack{\uparrow\\ \text{(bounded) linear operators}\\ \text{on } E}}{L(E)}$$

B – SODE to PDE, infinite dimensional 6a

For simplicity $A(x) = \sigma^T(x)\sigma(x) = A$ independent of $x \in E$. So, have:

$\sigma(X_t)$ above $\quad \in L(E)$, pos.

$$(1) \qquad \begin{aligned} \mathrm{d}X_t &= B(X_t)\mathrm{d}t + \sqrt{A}\,\mathrm{d}W_t \quad \longleftarrow \text{B.M. on } E \\ X_0 &= x \in E \quad = \text{sep. Hilbert space with } \langle\ ,\ \rangle \end{aligned}$$

Associated generator (Kolmogorov operator)

$$\begin{aligned} L\varphi(x) &= \sum_{i=1}^{N} \langle B(x)\,, e_i\rangle \frac{\partial \varphi}{\partial e_i}(x) + \frac{1}{2}\sum_{i,j=1}^{N} \langle A e_i, e_j\rangle \frac{\partial^2}{\partial e_i \partial e_j}\varphi(x) \\ &= \big\langle B(x)\,, D\varphi(x)\big\rangle + \frac{1}{2}\,\mathrm{Tr}(AD^2\varphi(x)) \end{aligned}$$

Fréchet derivatives

ONB of E

for $x \in E$ and $\varphi = g\big(\langle e_1, \cdot\rangle, \ldots, \langle e_N, \cdot\rangle\big) \longleftarrow$ all such $\mathcal{F}C_b^2$

$\in C_b^2(\mathbb{R}^N) \qquad \in \mathbb{N}$ arbitrary

Altogether:

$$(1)\qquad \begin{aligned} &\mathrm{d}X_t = B(X_t)\mathrm{d}t + \sqrt{A}\,\mathrm{d}W_t \\ &X_0 = x \in E \end{aligned}$$

Associated generator (Kolmogorov operator)

$$\begin{aligned} L\varphi(x) &= \sum_{i=1}^{N} \big\langle B(x)\,,e_i\big\rangle\,\frac{\partial\varphi}{\partial e_i}(x) + \frac{1}{2}\sum_{i,j=1}^{N}\langle Ae_i, e_j\rangle\,\frac{\partial^2}{\partial e_i \partial e_j}\varphi(x) \\ &= \big\langle B(x)\,,D\varphi(x)\big\rangle + \frac{1}{2}\,\mathrm{Tr}(AD^2\varphi(x)) \end{aligned}$$

for $x \in E$ and $\varphi = g\big(\langle e_1,\,\cdot\,\rangle, \ldots, \langle e_N,\,\cdot\,\rangle\big) : E \to \mathbb{R}$.

The associated heat equation is also called **Kolmogorov** (backward) **equation**

$$(2)\qquad \frac{\partial}{\partial t}u(t,x) = Lu(t,x), \quad u(0,\,\cdot\,) = f,$$

where $f : E \to \mathbb{R}$.

Want to solve (2), then (1)!

Once (2) is solved one has to apply highly developed machinery to get solution of (1).
In this talk we concentrate on solving (2):
Two approaches to solve (2) will be presented:

- L^p-approach
- Weighted function space (=WFS -) approach

C – Two stochastic PDE as examples

(a) Porous media equation (L^p-approach)

(b) Stochastic Navier–Stokes equation, $d = 2$ (WFS-approach)

(a) Stochastic porous media equation (L^p-approach)

with Dirichlet boundary conditions

$$dX_t = \underbrace{\Delta\Psi(X_t)}_{B(X_t)}\,dt + \sqrt{A}\,dW_t \qquad \text{with } \Psi : \mathbb{R} \to \mathbb{R}$$

trace class

on $E := H^{-1}(\Lambda)$, $\Lambda \subset \mathbb{R}^d$, open;
so,

$$L\varphi(x) = \langle \Delta\Psi(x), D\varphi(x)\rangle + \frac{1}{2}\mathrm{Tr}AD^2\varphi(x), \quad x \in H^{-1}(\Lambda), \quad \text{for } \varphi : H^{-1}(\Lambda) \to \mathbb{R}.$$

Remark:

(i) $A \equiv 0$: enormous literature.

(ii) $A \not\equiv 0$: first papers [Da Prato / R.: JEE '04], [Barbu / Bogachev/Da Prato/R.: JFA '06] Subsequently, many others. Mainly, on SPDE, not on Kolmogorov equations: Kim, Wu, Zhang, ...
Among most recent: Da Prato/R./Rosowski/Wang: Comm. P.D.E. '06], [Ren/R./Wang: BiBos-preprint '06].

(c) Stochastic Navier–Stokes equation, $d = 2$ (WFS-approach)

$$\mathrm{d}X_t = \Big[\overbrace{\underset{\substack{\uparrow\\ \text{viscosity}}}{\nu}\underbrace{\Delta_s}_{\substack{\uparrow\\ \text{Stokes-Laplacian}\\ \text{with Dirichlet}\\ \text{boundary conditions}}} X_t - \langle X_t, \underbrace{\nabla}_{\substack{\uparrow\\ \text{gradient}\\ \text{on } \mathbb{R}^2}}\rangle_{\mathbb{R}^2} X_t}^{B(X_t)}\Big]\,\mathrm{d}t + \underbrace{\sqrt{A}}_{\substack{\uparrow\\ \text{trace class or even}\\ \text{finite dim. range}}}\,\mathrm{d}W_t,$$

on

$$E := \left\{x \in L^2(\Lambda \to \mathbb{R}^2, \mathrm{d}x) \;\middle|\; \underbrace{\mathrm{div}\ x}_{\substack{\text{in the sense of}\\ \text{distributions}}} = 0\right\},$$

$\Lambda \subset \mathbb{R}^2$, open, bounded, $\partial\Lambda$ smooth;
so,

$$L\varphi(x) = \langle \nu\Delta_s x - \langle x, \nabla\rangle_{\mathbb{R}^2} x, D\varphi(x)\rangle_E, + \frac{1}{2}\mathrm{Tr}\ AD^2\varphi(x), \quad x \in E, \quad \text{for } \varphi : E \to \mathbb{R}.$$

Remark:

(i) $A \equiv 0$: OVERWHELMING literature

(ii) $A \not\equiv 0$: on SPDE: OVERWHELMING literature
$A \not\equiv 0$: on Kolmogorov equations: Da Prato/Debussche (also $d = 3$!), Barbu, Flandoli, Gozzi,...
WFS-approach: [R./Sobol: Ann. Prob. '06] for $d = 1$., [R./Sobol: Preprint '06] for $d = 2$ and also for geostrophic equation

(iii) Existence of infinitesimally invariant measures also proved for $d \geq 2$: [Bogachev / R.: PTRF '00]

D – Strategies to solve the Kolmogorov equation

to solve

$$(2) \qquad \frac{\partial}{\partial t} u(t,x) = Lu(t,x), \qquad u(0,\,\cdot\,) = f.$$

semigroup approach!

Construct

$$e^{tL} f(x) =: u(t,x), \qquad t \geq 0.$$

If e^{tL} exists, then by operator calculus

$$(\lambda - L)^{-1} = \int_0^\infty e^{-\lambda t} e^{tL} \, \mathrm{d}t\,,\ \lambda > \lambda_0.$$

So, try to construct $(\lambda - L)^{-1}$, $\lambda > \lambda_0$, and invert Laplace transform, (well-known method: “Hille-Yosida Theorem”).

For implementation two major steps necessary:

Step 1
Show "dissipativity", i.e.

$$\|(\lambda - L)\varphi\|_{W(E)} \geq (\lambda - \lambda_0)\|\varphi\|_{W(E)} \quad \forall \varphi \in \mathcal{F}C_b^2, \quad \lambda > \lambda_0,$$

for suitable norm $\|\cdot\|_{W(E)}$ in Banach space $W(E)$ of functions $f : E \to \mathbb{R}^d$ such that $\mathcal{F}C_b^2 \subset W(E)$. So, $\lambda - L$ is invertible for all $\lambda > \lambda_0$.

Step 2
Show "density of range", i.e. $(\lambda - L)(\mathcal{F}C_b^2)$ is dense in $(W(E), \|\cdot\|_{W(E)})$ for one (hence all) $\lambda > \lambda_0$. (Easier to achieve for weaker norms!)

Cannot take: $W = C_b(E)$, since coefficients of L not continuous in general.

In this talk:
Only Step 1 in

- L^p - approach for stochastic porous media equation.
  Here $W(E) := L^p(E, \mu)$ for suitable measures on E!
- WFS-approach for stochastic Navier-Stokes equation
  Here $W(E) :=$ weighted space of sequentially weakly continuous functions.

E - L^p -Approach

General idea of L^p-approach:

Step 1: Reference measures on E.

Solve $L^*\mu = 0$ "μ is L-infinitesimally invariant". (i.e. solve an elliptic problem first!)

i.e. find probability measure μ on $\overbrace{\mathcal{B}(E)}^{\text{Borel } \sigma\text{-algebra}}$ such that $L\varphi \in L^1(E,\mu)$ and

$$\int L\varphi \, \mathrm{d}\mu = 0 \quad \forall\, \varphi \in \mathcal{F}C_b^2 .$$

Then not hard to show: $(L, \mathcal{F}C_b^2)$ is dissipative on $L^p(E,\mu)$
(so has closure $\big(\bar{L}, D(\bar{L})\big)$ on $L^p(E,\mu)$ for all $p \in [1,\infty)$)

Step 2:

Show: $(\lambda - L)(\mathcal{F}C_b^2)$ dense in $L^p(E,\mu)$

Then $\exists\, e^{t\bar{L}}$, $t>0$, on $L^p(E,\mu)$ hence

$$L^p(E,\mu)\text{-}\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{e^{t\bar{L}} f}_{u(t,\,\cdot\,)} = \bar{L}(\underbrace{e^{t\bar{L}} f}_{u(t,\,\cdot\,)}), \quad t>0,\ f \in D(\bar{L}), \qquad \text{"solution in } L^p\text{"}$$

Remark. Then $\displaystyle\int e^{t\bar{L}} f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu \quad \forall\, t>0$ "μ invariant"

F – L^p-Approach for

Stochastic Porous Medium Equation

Now **Step 1** for stochastic porous medium equation (=SPME):

For simplicity $\Psi(x) = x^3$. So,

$$dX_t = \Delta(X_t^3)\,dt + \sqrt{A}\,\underset{\substack{\uparrow \\ =(W_t^i e_i)_{i\in\mathbb{N}} \\ \text{where } W_t^i \text{ indep. B. motions on } \mathbb{R}^1}}{dW_t} \qquad \text{(SPME)}$$

on $E := H^{-1}(\Lambda)$ $\big(:=$ dual of $\underset{\substack{\uparrow \\ \text{Dirichlet bd. cond.}}}{H_0^1}(\Lambda)\big)$, $\Lambda \subset \mathbb{R}^d$, open, bdd., $\partial\Lambda$ smooth.

Have

$$H_0^1(\Lambda) \subset L^2(\Lambda) \subset H^{-1}(\Lambda) \xrightarrow[\text{bijection}]{\Delta^{-1}} H_0^1(\Lambda).$$

- $\{e_i \,|\, i \in \mathbb{N}\}$ = eigenbasis of Dirichlet Laplacian on $H^{-1}(\Lambda)$.
- $A \in L(H^{-1}, H^{-1})$, $Ae_i = \lambda_i e_i$ (“diagonal”)
- $\lambda_i \geq 0 \;\forall\, i \in \mathbb{N}$, and $\displaystyle\sum_{i=1}^{\infty} \lambda_i < \infty$ (“trace class”).

In this case for $\varphi \in \mathcal{F}C_b^2(H^{-1})$

$$L\varphi(x) = \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i \, \frac{\partial^2}{\partial e_i^2}\varphi(x) + {}_{H^{-1}}\big\langle \Delta x^3 \, , \; D\varphi(x)\big\rangle_{H_0^1} \, ,$$

$x \in L^2(\Lambda)$ $\big(\subset H^{-1}(\Lambda)\big)$ s.th. $x^3 \in H_0^1$.
(So, can only be written in this form for *special* $x \in H^{-1}(\Lambda)$)

Step 1: Solve $L^*\mu = 0$.

Let $V_2 : H^{-1}(\Lambda) \to [0, \infty]$ "Lyapunov function"

$$V_2(x) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Lambda x^2(\xi)\,\mathrm{d}\xi, & x \in L^2(\Lambda), \\ +\infty, & \text{else.} \end{cases}$$

Then for $x \in L^2(\Lambda)$ $\big(\subset H^{-1}(\Lambda)\big)$ s.th. $x^2, x^3 \in H_0^1(\Lambda)$

$$LV_2(x) = \underbrace{\frac{1}{2}\sum_{i=1}^{\infty} \lambda_i \int_\Lambda e_i^2(\xi)\,\mathrm{d}\xi}_{=:C=\text{const.}} + \underbrace{{}_{H^{-1}}\langle \Delta x^3, x\rangle_{H_0^1}}_{\substack{=-\frac{3}{4}\int_\Lambda |\nabla x^2(\xi)|^2\,\mathrm{d}\xi \\ =:-\Theta_2(x)}} = \underbrace{C}_{\geq 0} - \underbrace{\Theta_2(x)}_{\geq 0}$$

Restrict to $x \in \mathrm{span}\{e_1, \ldots, e_N\}$

$$\underset{\substack{\uparrow \\ \text{operator} \\ \text{on } \mathbb{R}^N!}}{L_N} V_2(x) = \underbrace{\frac{1}{2}\sum_{i=1}^{N} \lambda_i \int_\Lambda e_i^2(\xi)\,\mathrm{d}\xi}_{\leq C} + \underbrace{{}_{H^{-1}}\big\langle P_N(\Delta x^3), x\big\rangle_{H_0^1}}_{=-\Theta_2(x)} \leq \underbrace{C - \Theta_2(x)}_{\substack{\text{independent} \\ \text{of } N!}}$$

Relatively easy to show: ([Bogachev / R.: Th. Prob. Appl. '00])

$\exists$ prob. measure μ_N on $\mathrm{span}\{e_1, \ldots, e_N\} \cong \mathbb{R}^N$ s.th. $L_N^* \mu_N = 0$, so

$$0 = \int L_N V_2 \,\mathrm{d}\mu_N \leq C - \int \Theta_2 \,\mathrm{d}\mu_N \qquad ❄$$

$$0 = \int L_N V_2 \, \mathrm{d}\mu_N \le C - \int \Theta_2 \, \mathrm{d}\mu_N \qquad ❄$$

Consider μ_N on $H^{-1}(\Lambda)$ $\big(\supset \operatorname{span}\{e_1, \dots, e_N\}\big)$, then

$$\sup_N \mu_N\big(\underbrace{\{\Theta_2 > R\}}_{\substack{\text{have compact}\\ \text{complements}\\ \text{in } H^{-1}(\Lambda)}}\big) \overset{\text{Chebychev}}{\le} \frac{1}{R} \cdot \sup_N \int \Theta_2 \, \mathrm{d}\mu_N \overset{❄}{\le} \frac{1}{R} \cdot C \xrightarrow{R \to \infty} 0$$

$$\overset{\text{Prokhorov}}{\Rightarrow} \exists\, \mu := \lim_{k\to\infty} \mu_{N_k} \qquad \text{in weak topology of measures on } H^{-1}(\Lambda)$$

$$\text{and} \quad \int \Theta_2 \, \mathrm{d}\mu \le C.$$

(Can show similarly: $\displaystyle\int |\nabla x^3|^2_{L^2} \, \mu(\mathrm{d}x) < \infty$,

so $\mu\Big(\big\{x \in L^2(\Lambda) \;\big|\; x^2, x^3 \in H_0^1(\Lambda)\big\}\Big) = 1.$)

Then show (again work!)

$$L^*\mu \quad \big(\overset{!}{=} \lim_{k\to\infty} L^*_{N_k} \mu_{N_k}\big) \quad = 0.$$

G – WFS -Approach

General idea of WFS-approach for

Step 1

Prove a weighted maximum principle in infinite dimension, i.e. show (in applications by finite dimensional approximation):
There exist two functions $\mathbb{V}, \mathbb{W} : E \to \mathbb{R}_+, \mathbb{V} \leq \mathbb{W}$ both with weakly compact levels sets $\{\mathbb{V} \leq R\}$, $\{\mathbb{W} \leq R\}$, $R > 0$, such that for some $\lambda_0 > 0$

$$\sup_{x \in \{\mathbb{W} < \infty\}} \frac{(\lambda_0 - L)u}{\mathbb{W}}(x) \geq \sup_{x \in \{\mathbb{V} < \infty\}} \frac{u}{\mathbb{V}}(x),$$

Then

(a variant of) $(L, \mathcal{F}C_0^2)$ is dissipative on $W(E)$,

where the Banach space $W(E)$ is defined by

$$W(E) := \Big\{u : \{\mathbb{V} < \infty\} \to \mathbb{R} \Big| f_{\restriction \{\mathbb{V} \leq R\}} \text{ is weakly continuous } \forall R > 0 \text{ and } \lim_{R \to \infty} \sup_{\{\mathbb{V} \geq R\}} \frac{|f|}{\mathbb{V}} = 0\Big\} =: C_{\mathbb{V}}$$

equipped with the norm

$$\|u\|_{W(E)} := \sup_{\{\mathbb{V} < \infty\}} \frac{|u|}{\mathbb{V}}.$$

H – WFS -Approach for Stochastic Navier-Stokes Equation

Step 1

Weighted maximum principle holds with $(\kappa, \alpha > 0,\ \kappa > \alpha(1+\nu^{-2}))$

$$\mathbb{V}(x) := e^{\kappa\|x\|_E^2}(1+\|\nabla x\|_E^2)^\alpha$$

and

$$\mathbb{W}(x) := \nu\mathbb{V}(x)(\kappa\|\nabla x\|_2^2 + \alpha\|\Delta x\|_2),$$

$$x \in E = \{x \in L^2(\Lambda \to \mathbb{R}^d, \mathrm{d}\xi) \mid \operatorname{div} x = 0\}.$$