High Energy limits of Dirac type eigenfunctions

A. Strohmaier (joint work with D. Jakobson)

DMV Tagung 2006

Motivation

Theorem (Shnirelman, C. de Verdiere, Zelditch)

Let X be a compact Riemannian manifold and let ϕ_i be an orthonormal sequence in $L^2(X)$ consisting of eigenfunctions. If the geodesic flow on T_1^*X is ergodic there is a subsequence ϕ_j' of counting density one such that

$$|\phi_j'(x)|^2 \rightarrow 1$$

in the weak topology of measures.
(Ergodicity implies Quantum Ergodicity)

Microlocal version

Theorem (Shnirelman, C. de Verdiere, Zelditch)

Let X be a compact Riemannian manifold and let ϕ_i be an orthonormal sequence in $L^2(X)$ consisting of eigenfunctions. If the geodesic flow on T_1^*X is ergodic there is a subsequence ϕ_j' of counting density one such that

$$\langle \phi_j', A\phi_j' \rangle \to \int_{T_1^*X} \sigma_A(\xi) dL(\xi)$$

for all $A \in \Psi\mathrm{DO}^0_{cl}(X)$.
(Ergodicity implies microlocal Quantum Ergodicity)

Bundle valued operators

- Question: what about bundle valued operators like the Laplace Beltrami operator or the Dirac operators on a spin manifold. The situation is slightly different:
- One cannot expect a direct analog to hold. Eg. coclosed and closed eigen-p-forms give rise to different quantum limits.
- the relevant algebra of pseudodifferential operators is $\Psi\mathrm{DO}^0_{cl}(X,E)$ quantum limits are the states on $\overline{\Psi\mathrm{DO}^0_{cl}(X,E)}/\mathcal{K} \cong C(T^*_1X, \pi^*(End(E)))$ which is a noncommutative C^*-algebra.

Bundle valued operators

- Question: what about bundle valued operators like the Laplace Beltrami operator or the Dirac operators on a spin manifold. The situation is slightly different:
- One cannot expect a direct analog to hold. Eg. coclosed and closed eigen-*p*-forms give rise to different quantum limits.
- the relevant algebra of pseudodifferential operators is $\overline{\Psi\mathrm{DO}^0_{cl}(X,E)}$ quantum limits are the states on $\overline{\Psi\mathrm{DO}^0_{cl}(X,E)}/\mathcal{K} \cong C(T^*_1 X, \pi^*(End(E)))$ which is a noncommutative C^*-algebra.

Bundle valued operators

- Question: what about bundle valued operators like the Laplace Beltrami operator or the Dirac operators on a spin manifold. The situation is slightly different:
- One cannot expect a direct analog to hold. Eg. coclosed and closed eigen-p-forms give rise to different quantum limits.
- the relevant algebra of pseudodifferential operators is $\Psi\mathrm{DO}^0_{cl}(X, E)$ quantum limits are the states on $\Psi\mathrm{DO}^0_{cl}(X, E)/\mathcal{K} \cong C(T^*_1 X, \pi^*(End(E)))$ which is a noncommutative C^*-algebra.

Ergodicity of frame flows

Let FX be the frame bundle and let $p : FX \to T_1^*X$ be the projection onto the first vector. The geodesic flow extends by parallel translation to a flow on FX, the frame flow. If X is negatively curved with sectional curvatures satisfying $-K_2^2 \leq K \leq -K_1^2$. The frame flow is known to be ergodic

- if X has constant curvature (Brin 76, Brin-Pesin 74);
- for an open and dense set of negatively curved metrics (in the C^3 topology) (Brin 75);
- if n is odd, but not equal to 7 (Brin-Gromov 80); or if $n = 7$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03);
- if n is even, but not equal to 8, and $K_1/K_2 > 0.93$, (Brin-Karcher 84); or if $n = 8$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03).

Ergodicity of frame flows

Let FX be the frame bundle and let $p : FX \to T_1^*X$ be the projection onto the first vector. The geodesic flow extends by parallel translation to a flow on FX, the frame flow. If X is negatively curved with sectional curvatures satisfying $-K_2^2 \leq K \leq -K_1^2$. The frame flow is known to be ergodic

- if X has constant curvature (Brin 76, Brin-Pesin 74);
- for an open and dense set of negatively curved metrics (in the C^3 topology) (Brin 75);
- if n is odd, but not equal to 7 (Brin-Gromov 80); or if $n = 7$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03);
- if n is even, but not equal to 8, and $K_1/K_2 > 0.93$, (Brin-Karcher 84); or if $n = 8$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03).

Ergodicity of frame flows

Let FX be the frame bundle and let $p : FX \to T_1^*X$ be the projection onto the first vector. The geodesic flow extends by parallel translation to a flow on FX, the frame flow. If X is negatively curved with sectional curvatures satisfying $-K_2^2 \leq K \leq -K_1^2$. The frame flow is known to be ergodic

- if X has constant curvature (Brin 76, Brin-Pesin 74);
- for an open and dense set of negatively curved metrics (in the C^3 topology) (Brin 75);
- if n is odd, but not equal to 7 (Brin-Gromov 80); or if $n = 7$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03);
- if n is even, but not equal to 8, and $K_1/K_2 > 0.93$, (Brin-Karcher 84); or if $n = 8$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03).

Ergodicity of frame flows

Let FX be the frame bundle and let $p : FX \to T_1^*X$ be the projection onto the first vector. The geodesic flow extends by parallel translation to a flow on FX, the frame flow. If X is negatively curved with sectional curvatures satisfying $-K_2^2 \leq K \leq -K_1^2$. The frame flow is known to be ergodic

- if X has constant curvature (Brin 76, Brin-Pesin 74);
- for an open and dense set of negatively curved metrics (in the C^3 topology) (Brin 75);
- if n is odd, but not equal to 7 (Brin-Gromov 80); or if $n = 7$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03);
- if n is even, but not equal to 8, and $K_1/K_2 > 0.93$, (Brin-Karcher 84); or if $n = 8$ and $K_1/K_2 > 0.99023...$ (Burns-Pollicot 03).

Quantum ergodicity for the p-form Laplacian

Consider the following restricted system on p-forms which form an ONB in $\ker(\delta)$.

$$\Delta_p \phi_j = \lambda_j \phi_j, \quad \delta \phi_j = 0.$$

Theorem (JS)

If $p \neq \frac{n-1}{2}$ and the $2\min(p, n-p)$-frame-flow is ergodic, then there is a density one subsequence ϕ'_k that converges to a state ω_∞ on $C(T_1^*X, \pi^* End(\Lambda^p X))$.

$$\omega_\infty(a) := \binom{n-1}{p}^{-1} \int_{T_1^*X} \mathrm{tr}\left(i(\xi) i^*(\xi) a(\xi)\right) dL(\xi),$$

Quantum ergodicity for the p-form Laplacian

If $p = \frac{n-1}{2}$ there is a further symmetry $\delta *$ and we need a further constraint $i^{p+1} \delta * \phi_k = \pm \sqrt{\lambda_k} \phi_k$. With this further constraint Quantum ergodicity holds!
This is the case in dimension 3 for 1-forms, i.e. for electrodynamics in the physical dimension and is due to circular polarizations.

Quantum ergodicity for the Dirac operator

Theorem (JS)

Let X be a spin manifold with spinor bundle S and Dirac operator D. Let ϕ_j be an ONB in the positive energy subspace of D in $L^2(X;S)$ of eigensections. Then, if the frame flow is ergodic there is a density one subsequence ϕ_j' that converges to a state ω_+ on $C(T_1^*X, \pi^* End(S))$.

$$\omega_+(a) = \frac{1}{2^{[\frac{n}{2}]}} \int_{T_1^*X} \mathrm{tr}((1+\gamma(\xi))\, a(\xi)) dL(\xi).$$

Remarks

- Since the high energy limit is noncommutative it is "quantum" and shows some new features. For example ergodic decompositions are not unique (may split Dirac into chiral parts as well).
- Conclusion of the theorem is not true for negatively curved Kähler manifolds even though the geodesic flow is ergodic (frame flow is not).
- Ergodic decomposition of the frame flow has a quantum counterparts. (no anomalies yet).

Remarks

- Since the high energy limit is noncommutative it is "quantum" and shows some new features. For example ergodic decompositions are not unique (may split Dirac into chiral parts as well).
- Conclusion of the theorem is not true for negatively curved Kähler manifolds even though the geodesic flow is ergodic (frame flow is not).
- Ergodic decomposition of the frame flow has a quantum counterparts. (no anomalies yet).

Remarks

- Since the high energy limit is noncommutative it is "quantum" and shows some new features. For example ergodic decompositions are not unique (may split Dirac into chiral parts as well).
- Conclusion of the theorem is not true for negatively curved Kähler manifolds even though the geodesic flow is ergodic (frame flow is not).
- Ergodic decomposition of the frame flow has a quantum counterparts. (no anomalies yet).