Dividends:

Modelling, Option Pricing, Portfolio Optimization

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Outline

- *Dividends: Basic remarks*
- *Dividend modelling in continuous-time models*
- *Option pricing with dividends*
- *Portfolio optimization in the presence of dividends*
- *Conclusion*

1. *Dividends: Basic remarks*

Facts and ideas:

- The stock price is the present value of all future dividend payments
- Modelling the dividend stream => Modelling of the stock price process
- Dividend payments affect the stock price (and not vice versa !)
    - "No" effect for the stock holder (receives the dividends)
    - Effect on option values => ? (depends on the type of the option !)

Main problems:

- Modelling of dividends and impact on the form of the stock price
- Consequences for option pricing, in particular for American and exotic options

2. *Dividend modelling in continuous-time models*

Assumption:

Discrete (random) dividends D_i are paid at times $t_i > t$:

(GE)

$$\begin{aligned} S_1(t) &= E_t\left[\sum_{i=1}^{\infty} e^{-r(t_i-t)} D_i\right] \\ &= E_t\left(e^{-r(t_1-t)} D_1\right) + E_t\left[\sum_{i=2}^{\infty} e^{-r(t_i-t)} D_i\right] \\ &= E_t\left(e^{-r(t_1-t)} D_1\right) + E_t\left(e^{-r(s-t)} S_1(s)\right) \end{aligned}$$

for $t < t_1 < s < t_2$

⇒ **Possibilities:** Modelling of ***dividend stream*** or ***recursive modelling***

Practitioner models:

Bos and Vandermark (2002), Bos, Gairat, Shepeleva (2003), Haug, Haug, Lewis (2003), …

Model 1: ***completely known future dividends*** (Escrowed Model)

(1) $$S^{(1)}(t) = \tilde{S}(t) + D_t(T)$$

with $$d\tilde{S}(t) = \tilde{S}(t)\left(rdt + \sigma dW(t)\right), \quad D_t(T) = \sum_{i: t < t_i < T} d_i e^{-r(t_i - t)}$$

$\Rightarrow S^{(1)}(T) = \tilde{S}(T)$ and log-normality of the final stock price

$\Rightarrow$ Black-Scholes formula for ***European*** call options with

$S^{(1)}(0)$ replaced by $\tilde{S}(0) = S^{(1)}(0) - D_0(T)$

Main problem:

- dividends are assumed to be known
- absolute volatility is too low (?)

Model 2: ***forward dividend model***

(2) $$S^{(2)}(t) = \hat{S}(t) - D(t)$$

with $$d\hat{S}(t) = \hat{S}(t)\left(r\,dt + \sigma\,dW(t)\right), \quad D(T) = \sum_{i:0<t_i\leq t} d_i e^{-r(t_i - t)}$$

Main motivation: Black-Scholes formula for valueing ***European*** calls with

(3) $$\left(S^{(2)}(T) - K\right)^+ = \left(\hat{S}(T) - D(T) - K\right)^+ =: \left(\hat{S}(T) - \hat{K}\right)^+.$$

- Somewhat artificial
- Stock price can become negative !
- In Models 1 and 2 binomial methods can be used for pricing American options

Model 3: ***GBM between dividends***

(3) $$dS^{(3)}(t) = S^{(3)}(t)\left(rdt + \sigma dW(t)\right) - \sum_i d_i \delta(t - t_1)$$

- Practitioners: ***"the correct one"*** …. (but no BS formula)
- Used to (approximately) derive *local implied volatilities* to equate model 1 and 3 call prices (then use BS formulae with different (!) volatilities)

Model 4: ***Decomposition of dividends***

(4) $$D_t(T) = \sum_{i:t<t_i\leq T} \frac{T-t_i}{T} d_i e^{-r(t_i - t)} + \sum_{i:t<t_i\leq T} \frac{t_i}{T} d_i e^{-r(t_i - t)} =: D_t^{near}(T) + D_t^{far}(T)$$

(5) $$C(t, S(t); K, T) = C_{BS}\left(t, S(t) - D_t^{near}(T); K + e^{rT} D_t^{far}(T), T\right),$$

i.e. one uses the Black-Scholes formula where

- the actual share price is corrected via subtracting the "***near dividend***"
- the strike is increased by the forward value of the "***far dividend***"

Model 5: *Dividend yields*

Approximate discrete dividends via a dividend yield, i.e. a continuous payment stream of

(6) $$\delta S(t)\,dt$$

$\Rightarrow$ Black-Scholes formula for ***European*** call options with

$S^{(1)}(0)$ replaced by $\tilde{S}(0) = S^{(1)}(0) - D_0(T)$

- text book approach
- only rough approximation for indices

Main problems

- all models aim for a ***suitable BS-formula variant***, not for a realistic dividend model
- dividends are assumed to be ***known***
- only consideration of ***European*** options (or, more serious, treated American options as European ones !)

3. *Option pricing with discrete dividends*

<u>New dividend modelling approach</u> (K., Rogers (2005))

a) ***Dividend announcement time equals dividend payment time***

Dividend payments of D_i at time t_i per share

$\Rightarrow$ Arbitrage pricing theory:

(7) $$S(t) = E_t\left(\sum_{t_m > t} \beta(t_m) D_m\right) / \beta(t)$$ ***Ex-dividend*** price at time t

with $\beta(t) = exp\left(-\int_0^0 r(s)\,ds\right)$ the discount factor and E a suitable pricing measure

Remarks:

i) The stock price is only finite if the dividend payments satisfy growth conditions.

ii) It is possible that some future dividends might be known.

iii) The classical Gordon growth model (Gordon 1967) is a special case of our model.

From now on:

(8) $$D_j = \lambda X(t_j)$$

with X an exponential Lévy process, $\lambda > 0$, and

(9) $$E(X(t))/X(0) = exp(\mu t)$$ for some $\mu < r$.

Dividend payment times: $t_m = mh$, $m = 1,2, \ldots$

$\Rightarrow$

(10) $$S(t) = \sum_{m \geq k} e^{-r(mh-t)} \lambda E_t(X(mh)) = \lambda X(t) \frac{e^{-(r-\mu)(kh-t)}}{1-e^{-(r-\mu)h}}, \quad t \in ((k-1)h, kh).$$

(11) $$S(h) = \lambda X(h)\left(\frac{1}{1-e^{-(r-\mu)h}} - 1\right) = S(h-) - \lambda X(h) = S(h-)e^{-(r-\mu)h}$$

Remark:

Note that the absolute dividend payment is random, the relative dividend is known !

Proposition 1. The time-0 price of a European call option with strike K and expiry $T \in \left(kh,(k+1)h\right)$ is given as

(12) $$e^{-rT} E\left[\left(S(0)e^{-k(r-\mu)h} e^{(r-\mu)T} X(T)/X(0) - K\right)^{+}\right].$$

In the special case of $X(t) = exp\left(\sigma W(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)$ we have the (BS-)formula

(13) $$\tilde{S}_k \Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

for the European call price with

(14) $$d_1 = \frac{ln\left(\frac{\tilde{S}_k}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad \tilde{S}_k = S(0)e^{-(r-\mu)kh}$$

Remarks.

i) Note the similarity (and difference) to the BS-formula of Model 1.

ii) In the Brownian case, the market remains complete

iii) (10) + (11): discounted stock price is a supermartingale under the pricing measure (a martingale between dividend payment times and only decreases after a dividend payment).

(Necessary) Variations

b) ***Dividends are announced in advance***

Still: dividends are paid at times h, $2h$, … .

New: amount that will be paid is announced at times εh, $(1+\varepsilon)h$, … , and equals

(15) $\theta X\left(\left(k+\varepsilon\right)h\right)$ with $0 < \varepsilon < 1$, θ a fixed positive and known constant

⇒ share price = *ex-dividend* price + present value of the next dividend payment

⇒ **announcem. of dividend ≅ payment of its pv at announcem. time**

⇒ Choose $\lambda = \theta e^{-r\left(1-\varepsilon\right)h}$ to obtain the ***ex-dividend share price*** as

(16) $$S^{ex}\left(t\right) = \lambda X\left(t\right)\frac{e^{-\left(r-\mu\right)\left(\left(k+1+\varepsilon\right)h-t\right)}}{1-e^{-\left(r-\mu\right)h}}$$

and the ***cum-dividend price*** as

(17) $$S^{cum}\left(t\right) = S^{ex}\left(t\right) + \lambda X\left(\left(k+\varepsilon\right)h\right)e^{r\left(t-\left(k+\varepsilon\right)h\right)} \text{ for } t \in \left(\left(k+\varepsilon\right)h, \left(k+1\right)h\right).$$

Remarks.

i) If the option matures in some interval of the form $(kh,(k+\varepsilon)h)$ then a simple variant of Proposition 1 is valid.

ii) For $T\in((k+\varepsilon)h,(k+1)h)$ the cum-dividend price of the stock involves the value of X at two different times.

Proposition 2. The time-0 price of a European call option with strike K and expiry $T\in((k+\varepsilon)h,(k+1)h)$ is given as

$$(18)\qquad e^{-rT}E\left[\left(\frac{\lambda X(T)e^{(r-\mu)((k+1+\varepsilon)h-T)}}{1-e^{-(r-\mu)h}}+\lambda X((k+\varepsilon)h)e^{r(T-(k+\varepsilon)h)}-K\right)^{+}\right].$$

Remark.

Even in the BS-case numerical methods are needed for a European call price !

c) ***Changing dividend policy*** (assump. of b)

Assume that at time $t = (k+\varepsilon)h$ the company announces a dividend payment of

$$\Delta' \neq \theta X\left((k+\varepsilon)h\right).$$

with $\Delta = \Delta' e^{-r(1-\varepsilon)h}$ we obtain

(19) $$S(t) = \lambda X(t)\frac{e^{-(r-\mu)\left((k+\varepsilon)h-t\right)}}{1-e^{-(r-\mu)h}} \qquad \text{for } t \in \left(kh, (k+\varepsilon)h\right),$$

(20) $$S^{ex}(t) = S(t-) - \Delta \qquad \text{for } t = (k+\varepsilon)h.$$

⇒ Model the change in the dividend policy as (for k=1)

(21) $$S^{ex}(t) = a\lambda X(t)\frac{e^{-(r-\mu)\left((1+\varepsilon)h-t\right)}}{1-e^{-(r-\mu)h}} \qquad \text{for } t = (1+\varepsilon)h\ ,$$

(22) $$S^{cum}(t) = S^{ex}(t) + \Delta e^{r\left(t-(k+\varepsilon)h\right)} \qquad \text{for } t \in \left((k+\varepsilon)h, (k+1)h\right),$$

and equate the two representations for the ex-dividend price, solve for a, use the results of b) scaled by this obtained factor a.

d) ***Further aspects and remarks***

- Pricing American options

Proposition 3: Geske, Whaley, Roll formula

Assume that a stock pays a dividend of D at time h. Then, the price of an American call with strike K and maturity $T > h$ on this stock is given by

$$\left(S(0) - De^{-rh}\right)\Phi(b_1) - (K-D)e^{-rh}\Phi(b_2) +$$

$$+\left(S(0) - De^{-rh}\right)\Phi\left(a_1, -b_1; -\sqrt{h/T}\right) - (K-D)e^{-rh}\Phi\left(a_2, -b_2; -\sqrt{h/T}\right)$$

with
$$a_1 = \frac{ln\left(\left(S(0)-De^{-rh}\right)/K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad a_2 = a_1 - \sigma\sqrt{T},$$

$$b_1 = \frac{ln\left(\left(S(0)-De^{-rh}\right)/S^*\right) + \left(r + \frac{1}{2}\sigma^2\right)h}{\sigma\sqrt{h}}, \qquad b_2 = b_1 - \sigma\sqrt{h},$$

$\Phi(a,b;\rho)$ the bivariate standard normal distribution with correlation coefficient ρ,
S^* the unique value with $C_{BS}(S^*, T-h) = S^* + D - K$ if this is finite and $S^* = +\infty$ else.

If the height of the dividend payment is not yet known then no explicit formula is available and numerical integration together with solving a nonlinear equation a number of times is needed

- Calibration of the parameters

- σ, h, r, a and λ are already discussed and should be no problem
- for obtaining μ:
  - first possibility: calibration via a suitable "Black-Scholes formula" (calibrate $\tilde{S}_k$ with European call prices and from that obtain μ)
  - second possibility: use the relation

    (23) $e^{-(r-\mu)h} S(t-) = S(t) = S(t-) - \Delta$, i.e. $1 - e^{-(r-\mu)h} = \dfrac{\Delta}{S(t-)}$

    exactly at the dividend payment time.

4. *Portfolio optimization in the presence of dividends*

Assumption: Classical Merton setting (one stock, one bond, e.g. log-utility)

$\Rightarrow$

Optimal portfolio process: $\pi(t) = \dfrac{b-r}{\sigma^2}$

Any changes due to the presence of dividends ?

Answer: No !

Why:

- Holder of the stock receives the dividends (=> into cash)
- Stock price drops by the dividend amount (=> stock portfolio falls)
- Additional cash from dividends has to be used to ***fill up*** the stock portfolio part

5. *Conclusion and further aspects*

We proposed a simple model for the price of a dividend paying asset with

- apart from the dividends, the price dynamics are simple and conventional;
- no difficulty in dealing with dividends which are announced before they are paid;
- the model is based on arbitrage-pricing principles, and is completely consistent;
- the standard BS-model is a special case (via a limiting argument)

Aspects for further studies

- incorporate the possibility of supply shocks, modelled as an independent log-Lévy process multiplying the stock price
- a random dividend ratio λ
- look at exotic options (interesting topics: barrier options, ….)
- ……