Shape Optimization for Elliptic PDEs

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Overview

- Introduction to shape optimization
- Analysing the shape Hessian
- well-posed $\longleftrightarrow$ ill-posed
- Solving the state equation
- General shape discretizations

Shape optimization

$$J(\Omega) = \int_\Omega j\big(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})\big) d\mathbf{x} \to \min,$$

where $\quad \mathcal{A}u = f \quad$ in Ω, $\qquad \mathcal{B}u = g \quad$ on Γ,

subject to
$$C_i(\Omega) = \int_\Omega h_i(\mathbf{x}) d\mathbf{x} = c_i, \qquad i = 1, 2, \ldots, m,$$
$$C_i(\Gamma) = \int_\Gamma h_i(\mathbf{x}) d\sigma_{\mathbf{x}} = c_i, \qquad i = m+1, m+2, \ldots, n.$$

security set $D \subseteq \mathbb{R}^2 : \Omega \subseteq D$

$$\left.\begin{array}{l} j : D \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \\ f, g, h_i : D \to \mathbb{R} \end{array}\right\} \text{ are sufficiently smooth functions}$$

Applications:
- optimal shape design: electromagnets, wings, walls, . . .
- identification of inclusions or obstacles
- computation of free (stationary) surfaces

Domain variations

▶ **Method of mappings/perturbation of identity:** (Murat/Simon)

For a smooth function $\mathbf{V}:\Omega\to\mathbb{R}^2$ consider the domain variation

$$\Omega_\varepsilon[\mathbf{V}] = \{\mathbf{x}+\varepsilon\mathbf{V}(\mathbf{x}) : \mathbf{x}\in\Omega\}.$$

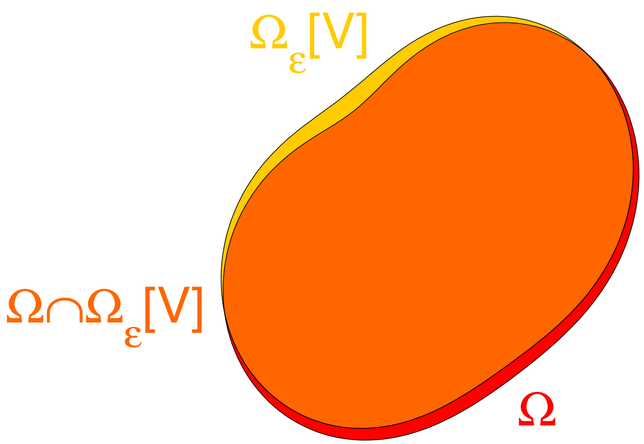
The directional derivative of the shape functional $J(\Omega)$ is defined by

$$\boxed{\nabla J(\Omega)[\mathbf{V}] = \lim_{\varepsilon\to 0}\frac{J(\Omega_\varepsilon[\mathbf{V}])-J(\Omega)}{\varepsilon}}$$

▶ **Local shape derivative du:** (Eppler/Kress/Potthast)

Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, & u &= g \quad \text{on } \Gamma,\\ -\Delta u_\varepsilon[\mathbf{V}] &= f \quad \text{in } \Omega_\varepsilon[\mathbf{V}], & u_\varepsilon[\mathbf{V}] &= g \quad \text{on } \Gamma_\varepsilon[\mathbf{V}]. \end{aligned}$$



The local shape derivative $du[\mathbf{V}]$, defined pointwise by

$$du(\mathbf{x})[\mathbf{V}] = \lim_{\varepsilon\to 0}\frac{u_\varepsilon[\mathbf{V}](\mathbf{x})-u(\mathbf{x})}{\varepsilon}, \quad \mathbf{x}\in\Omega\cap\Omega_\varepsilon[\mathbf{V}]$$

reads as

$$\boxed{\Delta du[\mathbf{V}] = 0 \quad \text{in } \Omega, \qquad du[\mathbf{V}] = \langle\mathbf{V},\mathbf{n}\rangle\frac{\partial(g-u)}{\partial\mathbf{n}} \quad \text{on } \Gamma.}$$

Computing the shape gradient

Consider the shape functional $J(\Omega) = \int_\Omega j(u,\mathbf{x})d\mathbf{x}$ where

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma.$$

The gradient requires only functions on the boundary

$$\begin{aligned}\nabla J(\Omega)[\mathbf{V}] &= \int_\Gamma \langle \mathbf{V},\mathbf{n}\rangle j(g,\mathbf{x})d\sigma_{\mathbf{x}} + \int_\Omega \underbrace{\frac{\partial j}{\partial u}(u,\mathbf{x})}_{-\Delta p} du[\mathbf{V}]d\mathbf{x} \\ &= \int_\Gamma \langle \mathbf{V},\mathbf{n}\rangle \left\{ j(g,\mathbf{x}) + \frac{\partial p}{\partial \mathbf{n}}\frac{\partial(g-u)}{\partial \mathbf{n}} \right\} d\sigma_{\mathbf{x}},\end{aligned}$$

since the local shape derivative satisifies

$$\Delta du[\mathbf{V}] = 0 \quad \text{in } \Omega, \qquad du[\mathbf{V}] = \langle \mathbf{V},\mathbf{n}\rangle \frac{\partial(g-u)}{\partial \mathbf{n}} \quad \text{on } \Gamma$$

and the adjoint state function p is defined by

$$-\Delta p = \frac{\partial j}{\partial u}(u,\mathbf{x}) \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma.$$

Theorem: (Hadamard) Provided that the shape functional $J(\Omega)$ is shape differentiable, the directional derivative $\nabla J(\Omega)[\mathbf{V}]$ is a scalar quantity defined on the boundary Γ.

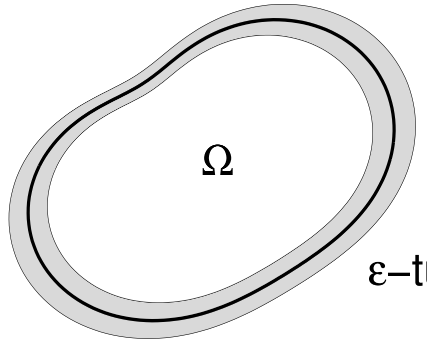
Second order shape calculus

- assume a fixed reference manifold $\widehat{\Gamma}$ with normal $\widehat{\mathbf{n}}$
- Ω represented via

$$\gamma(\mathbf{x}) = \mathbf{x} + r(\mathbf{x})\widehat{\mathbf{n}}(\mathbf{x}), \qquad \mathbf{x} \in \widehat{\Gamma}$$

- fixed variation field $\mathbf{V}(\mathbf{x}) = dr(\mathbf{x}) \cdot \widehat{\mathbf{n}}(\mathbf{x})$, i.e., Ω_ε represented via

$$\gamma_\varepsilon(\mathbf{x}) = \mathbf{x} + r(\mathbf{x})\widehat{\mathbf{n}}(\mathbf{x}) + \varepsilon dr(\mathbf{x})\widehat{\mathbf{n}}(\mathbf{x}), \qquad \mathbf{x} \in \widehat{\Gamma}$$



$\rightsquigarrow$ domain and perturbations are elements of some Banach space X

funcional analytical setting:

$$J : X \to \mathbb{R}$$
$$\nabla J : X \to X^\star$$
$$\nabla^2 J : X \to \mathcal{L}(X, X^\star)$$

Example: starlike domains

- $\widehat{\Gamma} = \mathbb{S} \quad \rightsquigarrow \quad \widehat{\mathbf{n}}(\phi) = \big(\cos(\phi), \sin(\phi)\big)^T, \quad \Gamma : \gamma(\phi) = r(\phi)\widehat{\mathbf{n}}(\phi)$
- we can identify the domain with a radial function $r \in C^{2,\alpha}_{\text{per}}([0, 2\pi])$,
- we can use fixed variation fields where $dr \in C^{2,\alpha}_{\text{per}}([0, 2\pi])$,
- both, the shape and its increment, are elements of the Banach space $C^{2,\alpha}_{\text{per}}([0, 2\pi])$.

Computing the shape Hessian

Assume that the domain $\Omega \in C^{2,\alpha}$ is starlike. Then:

$$\begin{aligned}\nabla^2 J(r)[dr_1, dr_2] = \int_0^{2\pi} & dr_1 dr_2 \left\{ j(u,\mathbf{x})\big|_\Gamma + \frac{\partial p}{\partial \mathbf{n}} \frac{\partial (g-u)}{\partial \mathbf{n}} \right\} \\ & + r\, dr_1 dr_2 \frac{\partial}{\partial \widehat{\mathbf{n}}} \left\{ j(u,\mathbf{x})\big|_\Gamma + \frac{\partial p}{\partial \mathbf{n}} \cdot \frac{\partial (g-u)}{\partial \mathbf{n}} \right\} \\ & + r\, dr_1 \left\{ \frac{\partial p}{\partial \mathbf{n}} \frac{\partial du[dr_2]}{\partial \mathbf{n}} + \frac{\partial dp[dr_2]}{\partial \mathbf{n}} \frac{\partial (g-u)}{\partial \mathbf{n}} \right\} d\phi,\end{aligned}$$

where the local shape derivatives $du[dr_2]$ and $dp[dr_2]$ solve

$$\begin{aligned} \Delta du[dr_2] &= 0 && \text{in } \Omega, \qquad du[dr_2] = +dr_2 \langle \widehat{\mathbf{n}}, \mathbf{n} \rangle \frac{\partial (g-u)}{\partial \mathbf{n}} && \text{on } \Gamma, \\ -\Delta dp[dr_2] &= du[dr_2] \frac{\partial^2 j}{\partial u^2}(u,\mathbf{x}) && \text{in } \Omega, \qquad dp[dr_2] = -dr_2 \langle \widehat{\mathbf{n}}, \mathbf{n} \rangle \frac{\partial p}{\partial \mathbf{n}} && \text{on } \Gamma. \end{aligned}$$

The shape Hessian defines a continuous bilinear from on $H^{1/2}([0,2\pi]) \times H^{1/2}([0,2\pi])$

$$\left| \nabla^2 J(r)[dr_1, dr_2] \right| \lesssim \|dr_1\|_{H^{1/2}([0,2\pi])} \|dr_2\|_{H^{1/2}([0,2\pi])}.$$

Nonlinear Ritz–Galerkin method

Consider finite dimensional ansatz space

$$V_N = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_N\} \subset C^{2,\alpha}_{\text{per}}([0,2\pi])$$

and seek $r_N^\star = \sum_{i=1}^N r_i\,\varphi_i \in V_N$ such that

$$\big(\nabla J(r_N^\star)[dr_1], dr_2\big)_{L^2([0,2\pi])} = 0 \quad \text{for all} \quad dr_1, dr_2 \in V_N.$$

Theorem: (Eppler/H./Schneider) Let the remainder estimate

$$\begin{aligned}\big|\nabla^2 J(r)[dr_1, dr_2] - \nabla^2 J(r^\star)[dr_1, dr_2]\big| \\ \le \eta\big(\|r - r^\star\|_{C^{2,\alpha}_{\text{per}}([0,2\pi])}\big)\|dr_1\|_{H^{1/2}([0,2\pi])}\|dr_2\|_{H^{1/2}([0,2\pi])}\end{aligned}$$

hold for all $dr_1, dr_2 \in H^{1/2}([0,2\pi])$. If $\nabla^2 J(r^\star)$ is strictly coercive

$$\nabla^2 J(r^\star)[dr, dr] \gtrsim \|dr\|^2_{H^{1/2}([0,2\pi])},$$

at the optimal domain $r^\star$, then we get the error estimate

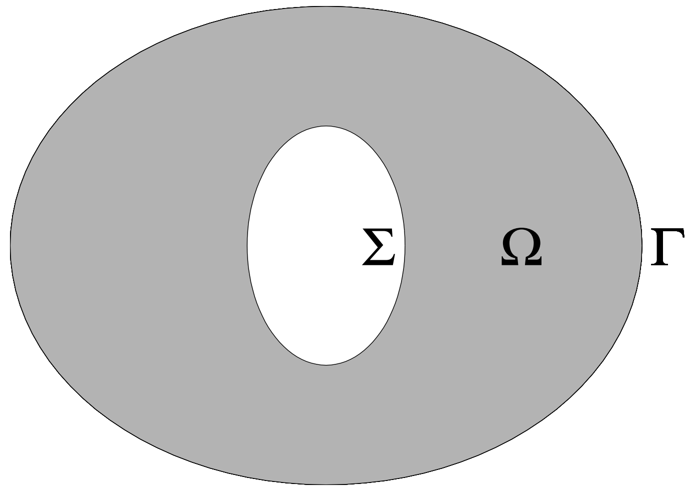
$$\big\|r^\star - r_N^\star\big\|_{H^{1/2}([0,2\pi])} \lesssim \inf_{v_N \in V_N} \big\|r^\star - v_N\big\|_{H^{1/2}([0,2\pi])}.$$

Free boundary problems

Given: Dirichlet data g on Σ and Neumann data h on Γ
(Bernoulli free boundary problem)

Problem: seek Γ such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega,\\ u &= g && \text{on } \Sigma,\\ u = 0, \quad -\frac{\partial u}{\partial \mathbf{n}} &= h && \text{on } \Gamma. \end{aligned}$$



Shape problem:

$$J(\Omega) = \int_\Omega \left\{ \|\nabla u\|^2 - 2fu + h^2 \right\} d\mathbf{x} \to \min,$$

$$\text{where} \qquad \begin{aligned} -\Delta u &= f && \text{in } \Omega,\\ u &= g && \text{on } \Sigma,\\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Shape gradient:

$$\nabla J(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V}, \mathbf{n} \rangle \left\{ h^2 - \|\nabla u\|^2 \right\} d\sigma_{\mathbf{x}} = \int_\Gamma \langle \mathbf{V}, \mathbf{n} \rangle \left\{ h^2 - \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 \right\} d\sigma_{\mathbf{x}}$$

Free boundary problems

Theorem: (Eppler/H.) The Hessian is strictly $H^{1/2}([0,2\pi])$-coercive if

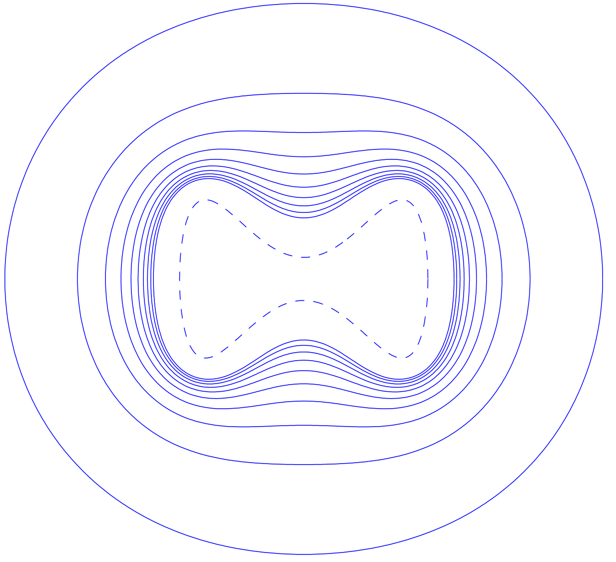
$$\kappa + \left\{\frac{\partial h}{\partial \mathbf{n}} - f\right\} \Big/ h \geq 0 \qquad \text{on } \Gamma^\star.$$

Especially, in the case $h \equiv const.$ and $f \equiv 0$, the shape Hessian is $H^{1/2}([0,2\pi])$-coercive if the boundary $\Gamma^\star$ is convex (seen from inside).

Proof: $$\nabla^2 J(\Omega^\star)[dr_1, dr_2] = \int_{\Gamma^\star} (Mdr_1) \left\{ D2N + \kappa + \left[\frac{\partial g}{\partial \mathbf{n}} - f\right] \Big/ g \right\} (Mdr_2) d\sigma$$

Examples:

- growth of anodes: $f \equiv 0$, $g \equiv 1$, and $h \equiv const.$
- electromagnetic shaping (2D): exterior boundary value problem, uniqueness ensured by volume constraint
- maximization of the torsional stiffness of a elastic cylindrical bar under simultaneous constraints on its volume and bending rigidity



Compactly supported shape functionals

Consider the shape problem $J(\Omega) = \int_B j(u, \mathbf{x}) d\mathbf{x} \to \min$ where

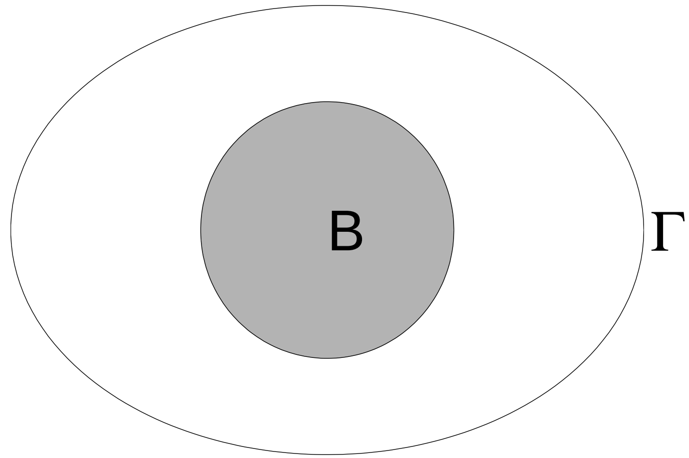
$-\Delta u = f$ in Ω, $u = g$ on Γ.

The gradient reads as

$$\nabla J(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V}, \mathbf{n} \rangle \frac{\partial p}{\partial \mathbf{n}} \frac{\partial (u-g)}{\partial \mathbf{n}} d\sigma_{\mathbf{x}},$$

where the adjoint state function p satisfies

$$-\Delta p = \frac{\partial j}{\partial u}(u, \mathbf{x})\Big|_B \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma.$$



The shape Hessian simplifies at the optimal domain $r^\star$ to

$$\nabla^2 J(r^\star)[dr_1, dr_2] = \int_0^{2\pi} r^\star \, dr_1 \, \frac{\partial dp[dr_2]}{\partial \mathbf{n}} \frac{\partial (g-u)}{\partial \mathbf{n}} d\phi,$$

where

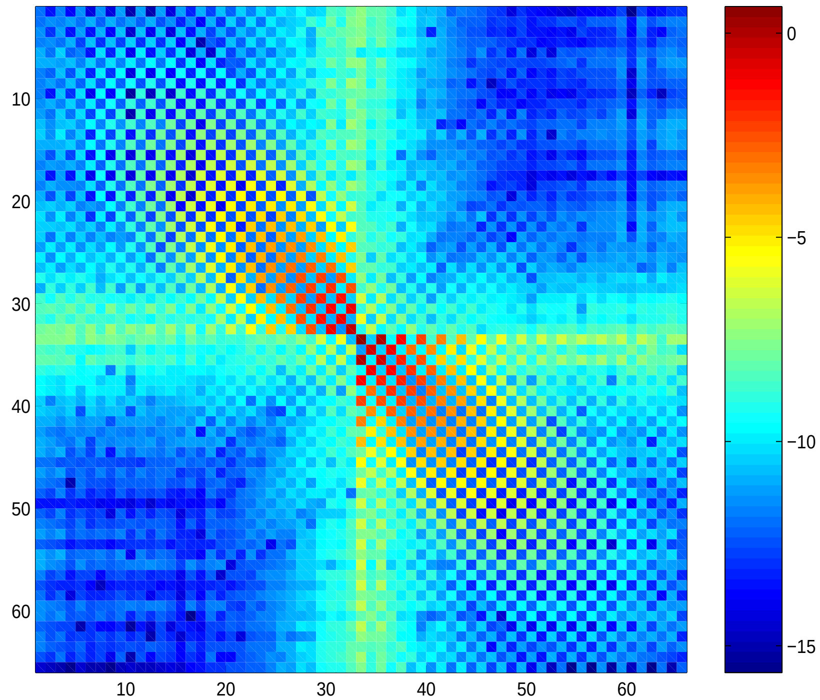
$$dp[dr_2](\mathbf{x}) = -\frac{1}{2\pi} \int_B \log \|\mathbf{x} - \mathbf{y}\| du[dr_2](\mathbf{y}) \frac{\partial^2 j}{\partial u^2}\big(u(\mathbf{y}), \mathbf{y}\big) d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \Omega^\star.$$

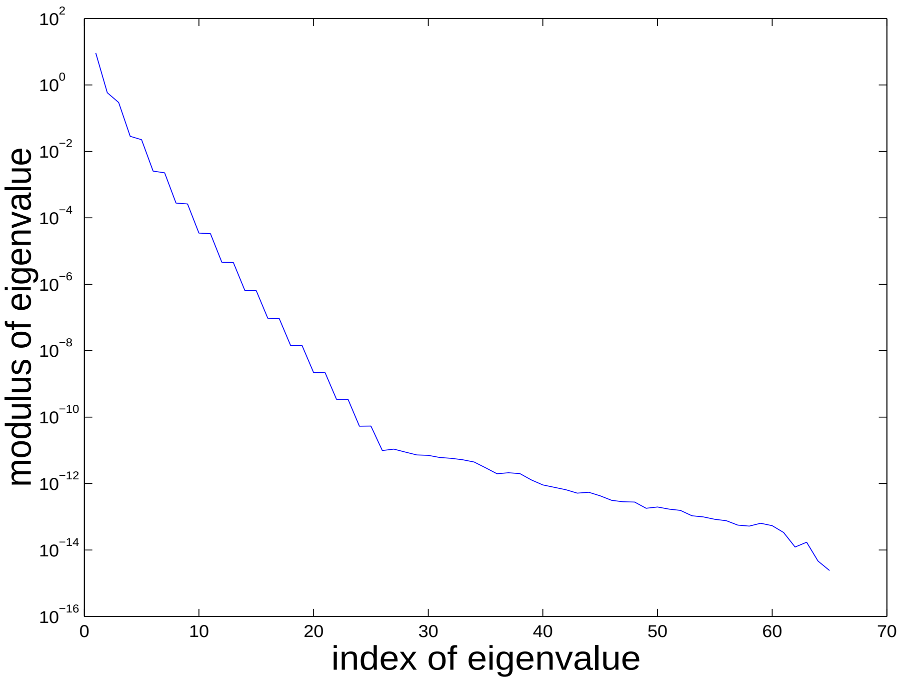
$\rightsquigarrow$ severely ill-posed problem: Hessian is arbitrarily compact!

$L^2(B)$-Tracking type functionals

Consider the shape problem $J(\Omega) = \int_B (u - u_d)^2 d\mathbf{x} \to \min$ where

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma.$$





Solving the state equation

- only boundary updates are practicable
- it suffices to consider boundary variations
- boundary integral representations of shape gradient & Hessian
- one has to compute the following quantities

$$\nabla J(\Omega): \ \frac{\partial u}{\partial \mathbf{n}}, \frac{\partial p}{\partial \mathbf{n}} \qquad \nabla^2 J(\Omega): \ \frac{\partial du[\mathbf{V}]}{\partial \mathbf{n}}, \frac{\partial dp[\mathbf{V}]}{\partial \mathbf{n}}, \frac{\partial^2 u}{\partial \mathbf{n}^2}, \frac{\partial^2 p}{\partial \mathbf{n}^2}, \frac{\partial^2 u}{\partial \mathbf{n}\partial \mathbf{t}}, \frac{\partial^2 p}{\partial \mathbf{n}\partial \mathbf{t}}$$

- BIE formulation seems to be very attractive if we consider the shape functional

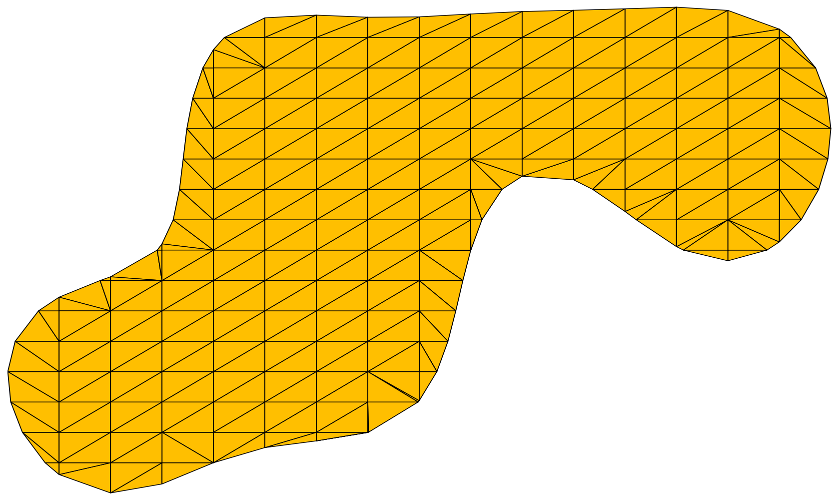
$$J(\Omega) = \int_\Omega \{h(\mathbf{x})u(\mathbf{x}) + h_0(\mathbf{x})\}d\mathbf{x} = \int_\Omega \{N_h f + h_0\}d\mathbf{x} + \int_\Gamma \left\{ g\frac{\partial N_h}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} N_h \right\} d\sigma_{\mathbf{x}}$$

with the state function

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma,$$

since the adjoint state function satisfies

$$-\Delta p = h \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma.$$



BIE formulation

Problem: $\boxed{\text{seek } u \text{ such that} \quad -\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma}$

There holds $u = v + N_f$, where N_f is a Newton potential, i.e. $-\Delta N_f = f$, and v satisfies

$$\Delta v = 0 \quad \text{in } \Omega, \qquad v = g - N_f \quad \text{on } \Gamma.$$

$$\boxed{\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} + \frac{\partial N_f}{\partial \mathbf{n}} \quad \text{with} \quad \mathcal{V}\frac{\partial v}{\partial \mathbf{n}} = \left(\frac{1}{2} + \mathcal{K}\right)v,}$$

where
$$(\mathcal{V}\rho)(\mathbf{x}) := -\frac{1}{2\pi}\int_\Gamma \log\|\mathbf{x}-\mathbf{y}\|\rho(\mathbf{y})d\sigma_{\mathbf{y}},$$
$$(\mathcal{K}\rho)(\mathbf{x}) := +\frac{1}{2\pi}\int_\Gamma \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_{\mathbf{y}}\rangle}{\|\mathbf{x}-\mathbf{y}\|^2}\rho(\mathbf{y})d\sigma_{\mathbf{y}}.$$

$$\frac{\partial^2 u}{\partial \mathbf{t}^2} = \frac{\partial^2 g}{\partial \mathbf{t}^2} - \kappa\left\{\frac{\partial v}{\partial \mathbf{n}} + \frac{\partial(N_f - g)}{\partial \mathbf{n}}\right\}, \qquad \frac{\partial^2 u}{\partial \mathbf{n}^2} = \kappa\left\{\frac{\partial v}{\partial \mathbf{n}} + \frac{\partial(N_f - g)}{\partial \mathbf{n}}\right\} - \frac{\partial^2 g}{\partial \mathbf{t}^2} + f,$$

$$\text{and} \quad \frac{\partial^2 u}{\partial \mathbf{n}\partial \mathbf{t}} = \frac{\partial^2 v}{\partial \mathbf{n}\partial \mathbf{t}} + \frac{\partial^2 N_f}{\partial \mathbf{n}\partial \mathbf{t}} \quad \text{with}$$

$$\mathcal{V}\frac{\partial^2 v}{\partial \mathbf{n}\partial \mathbf{t}} = \left(\frac{1}{2} + \mathcal{K}\right)\frac{\partial(g - N_f)}{\partial \mathbf{t}} + \left[\frac{1}{2} + \mathcal{K}, \frac{\partial}{\partial \mathbf{t}}\right](g - N_f) - \left[\mathcal{V}, \frac{\partial}{\partial \mathbf{t}}\right]\frac{\partial v}{\partial \mathbf{n}}.$$

BEM formulation

$$\boxed{\text{Dirichlet-to-Neumann map:} \quad \mathcal{V}\frac{\partial v}{\partial \mathbf{n}} = \left(\frac{1}{2} + \mathcal{K}\right) v}$$

▶ **Traditional Galerkin scheme:** ansatz: $\dfrac{\partial v}{\partial \mathbf{n}} = \Phi^{(1)}\sigma$

$\rightsquigarrow$ Galerkin system: $\boxed{\mathbf{V}\sigma = (\tfrac{1}{2}\mathbf{B} + \mathbf{K})\mathbf{G}^{-1}\mathbf{v}}$

where

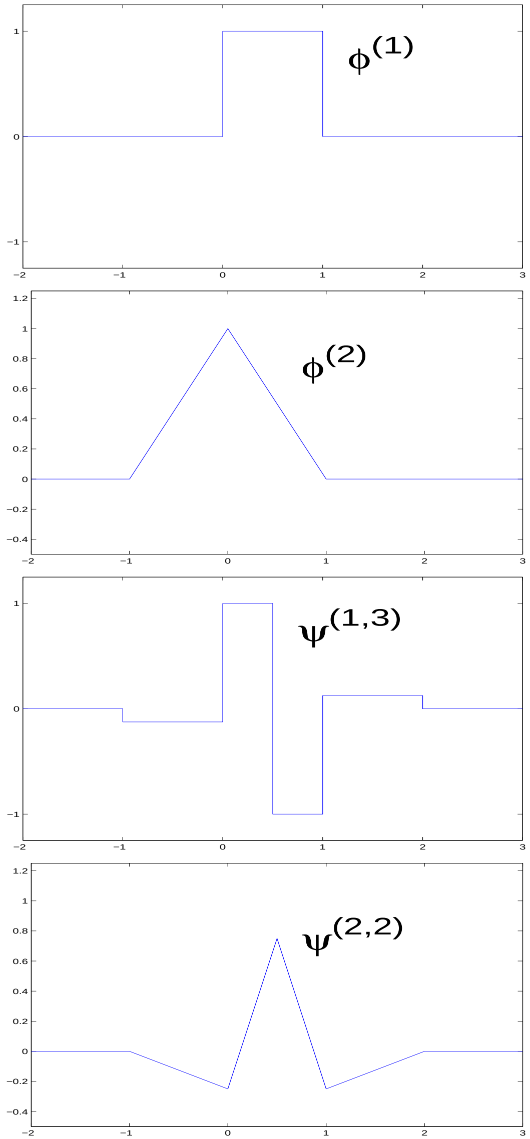
$$\mathbf{V} = (\mathcal{V}\Phi^{(1)}, \Phi^{(1)})_{L^2(\Gamma)}, \quad \mathbf{B} = (\Phi^{(2)}, \Phi^{(1)})_{L^2(\Gamma)},$$
$$\mathbf{K} = (\mathcal{K}\Phi^{(2)}, \Phi^{(1)})_{L^2(\Gamma)}, \quad \mathbf{G} = (\Phi^{(2)}, \Phi^{(2)})_{L^2(\Gamma)},$$
$$\mathbf{v} = (v, \Phi^{(2)})_{L^2(\Gamma)}.$$

$$\left.\begin{array}{l} \bullet \text{ densely populated system matrices} \\ \bullet \text{ ill conditioned } \operatorname{cond}(\mathbf{V}) \sim N_\Gamma \end{array}\right\} \longrightarrow \text{complextity } O(N_\Gamma^2)$$

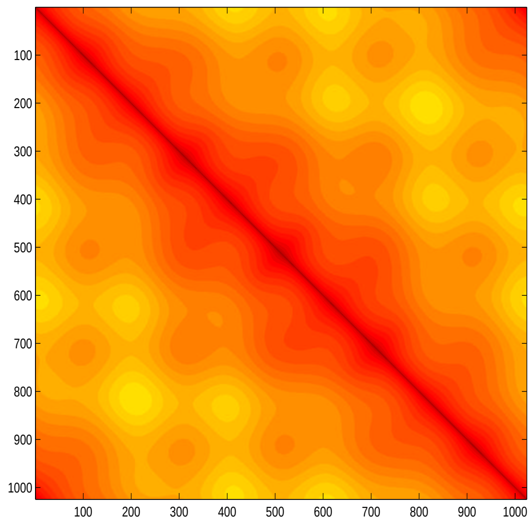
▶ **Wavelet Galerkin scheme:**

(Beylkin/Coifman/Rokhlin, Dahmen/Prößdorf/Schneider)

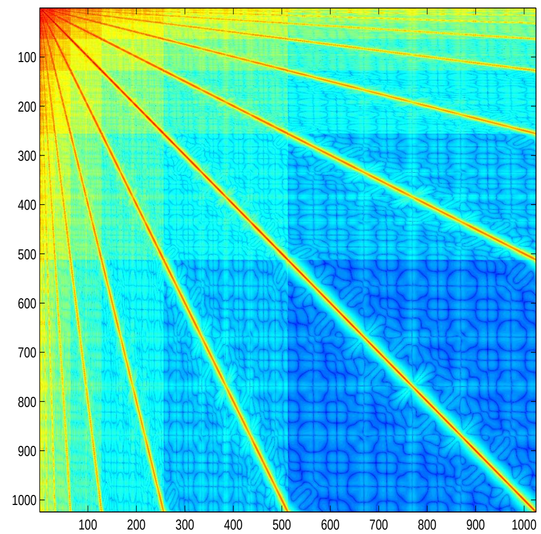
$$\left.\begin{array}{l} \bullet \text{ sparse matrix approximation} \\ \bullet \operatorname{diag}(\mathbf{V}) \text{ preconditioner to } \mathbf{V} \end{array}\right\} \longrightarrow \text{complextity } O(N_\Gamma)$$



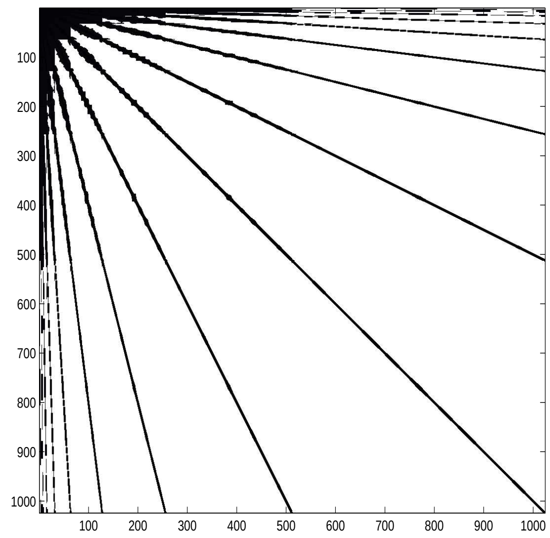
Wavelet Galerkin scheme



single-scale basis



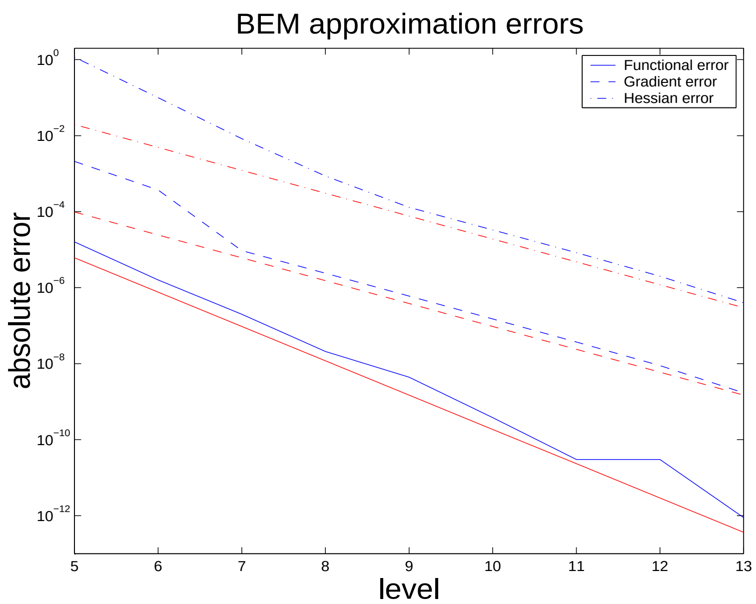
wavelet basis



compression

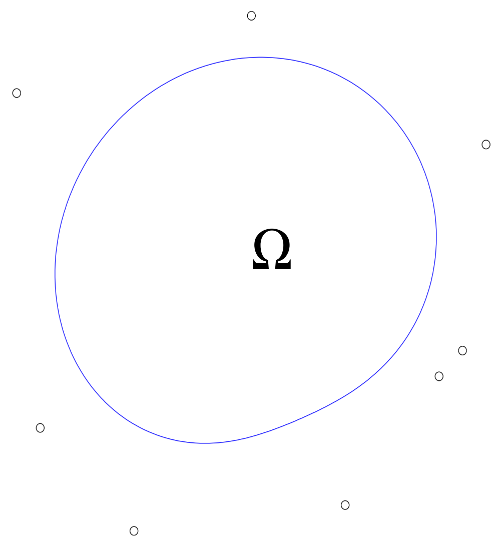
Approximation errors of the functional data:

$$|J(\Omega) - J(\Omega)| = O(h^3)$$
$$\|\nabla J(\Omega) - \nabla J(\Omega)\| = O(h^2)$$
$$\|\nabla^2 J(\Omega) - \nabla^2 J(\Omega)\| = O(h^2)$$



Electromagnetic shaping (2D)

Given: cylindrical vertical column of liquid metal with cross section $\Omega^c \in \mathbb{R}^2$ and M vertical circular conductors with radii ε



Shape problem: (Pierre/Henrot/Roche/Sokolowski/Novruzi)

$$J(\Omega) = -\int_{\Omega^c} \|\nabla u\|^2 d\mathbf{x} + A \int_\Gamma 1 d\sigma_{\mathbf{x}} \to \min,$$

where
$$\begin{aligned} -\Delta u = j := \textstyle\sum_{i=1}^M \frac{\alpha_i I}{\pi\varepsilon^2}\chi_{B_\varepsilon(\mathbf{x}_i)} & \quad \text{in } \Omega^c, \\ u = 0 & \quad \text{on } \Gamma, \\ u = O(1) & \quad \text{as } \|\mathbf{x}\| \to \infty, \\ \nabla u = O(\|\mathbf{x}\|^{-2}) & \quad \text{as } \|\mathbf{x}\| \to \infty. \end{aligned}$$

subject to $|\Omega| = V_0$.

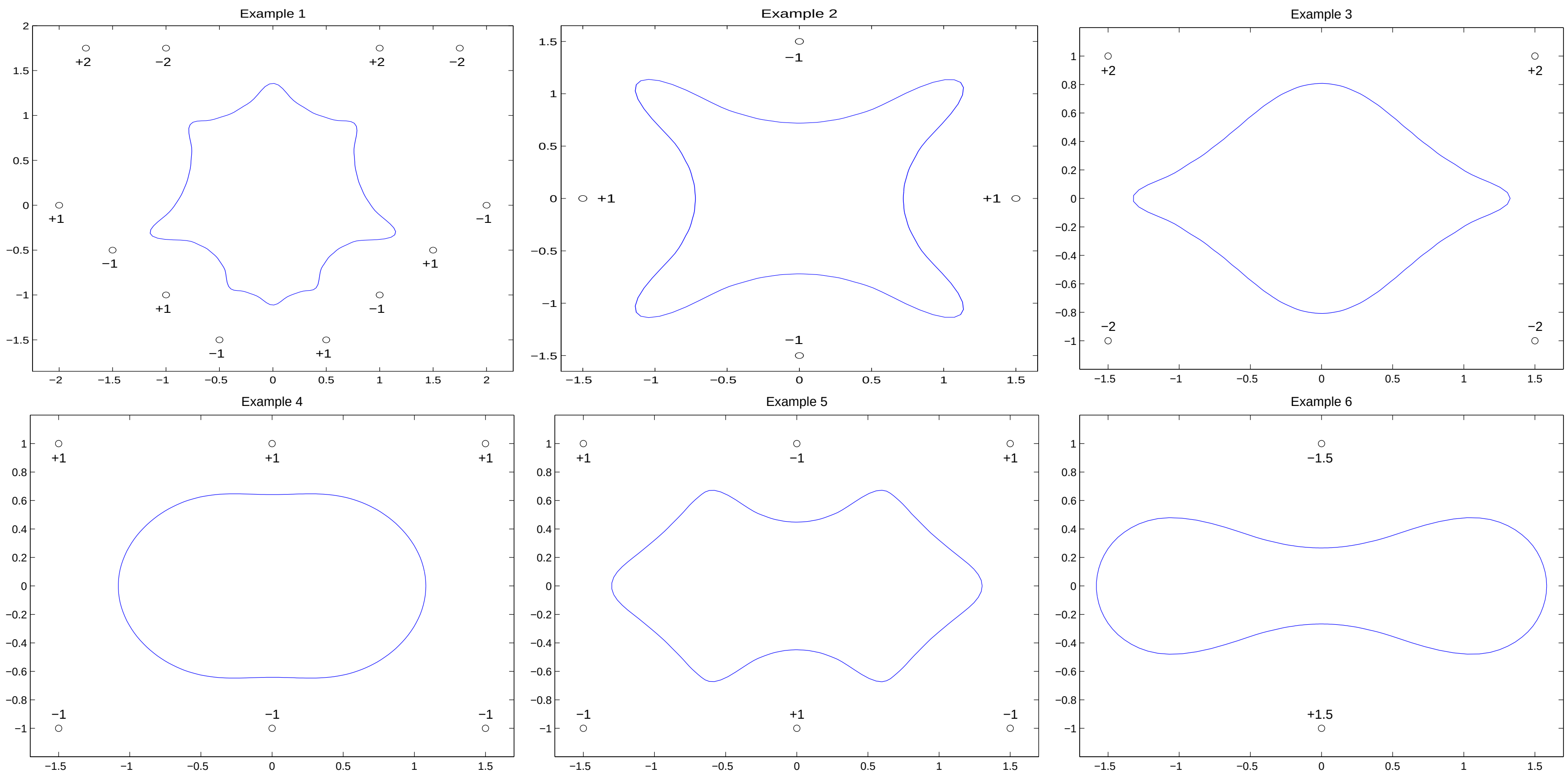
Shape gradient:

$$\nabla J(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V}, \mathbf{n}\rangle \left\{ A\kappa + \left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 \right\} d\sigma_{\mathbf{x}}$$

$\leadsto$ physical equilibrium condition: $A\kappa + \left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 = p \equiv const.$ on $\Gamma^\star$

Electromagnetic shaping (2D)

Example	iterations	$\left\|\nabla L_\alpha^{(n)}\left(\left(\Omega^{(n)},\lambda(\Omega^{(n)})\right)\right)\right\|$	λ	cpu-time
1	11	1.171e-07	3.010e-02	26 sec.
2	15	1.026e-09	7.575e-02	36 sec.
3	11	3.960e-07	3.539e-01	27 sec.
4	11	2.659e-08	9.738e-01	30 sec.
5	15	3.649e-07	8.851e-02	36 sec.
6	15	2.055e-09	2.191e-01	36 sec.



Compactly supported shape functionals

Consider the shape problem $J(\Omega) = \int_B j(u,\mathbf{x})d\mathbf{x} \to \min$ where

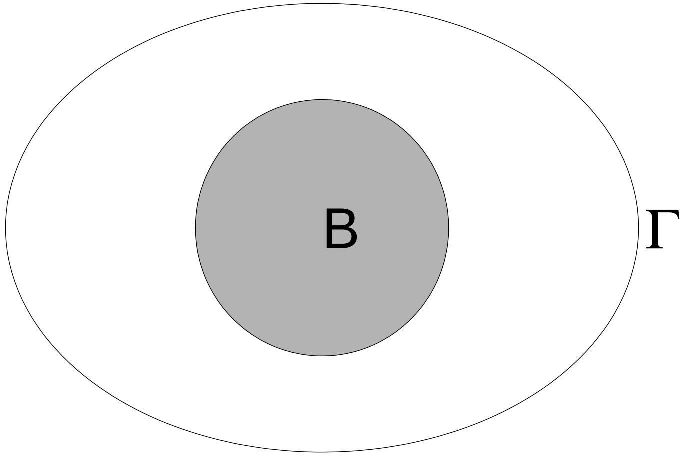
$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma.$$

The gradient reads as

$$\nabla J(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V},\mathbf{n}\rangle \frac{\partial p}{\partial \mathbf{n}} \frac{\partial(u-g)}{\partial \mathbf{n}} d\sigma_{\mathbf{x}},$$

where the adjoint state function p satisfies

$$-\Delta p = \frac{\partial j}{\partial u}(u,\mathbf{x})\Big|_B \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma.$$



Consequence:

we need fast access to u on the compact set B

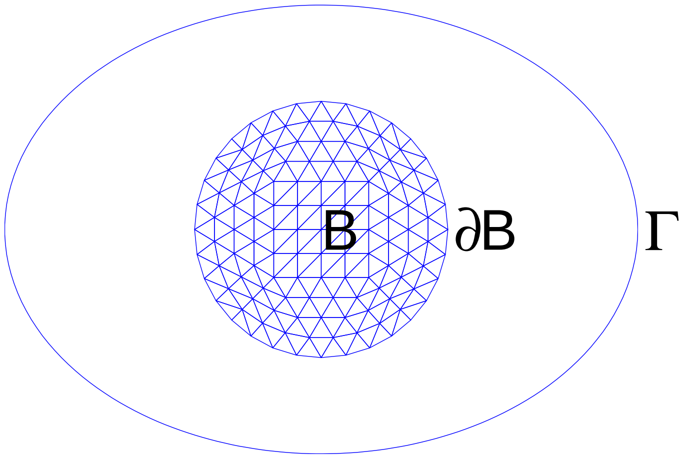
$\rightsquigarrow$ coupling of FEM and BEM

Coupling of FEM-BEM

Two integral formulation:

Find $(u, \partial u/\partial \mathbf{n})$ such that

$$
\begin{aligned}
-\triangle u &= f && \text{in } B,\\
-\mathcal{W}u + \Big(\frac{1}{2} - \mathcal{K}^\star\Big)\frac{\partial u}{\partial \mathbf{n}} &= \frac{\partial u}{\partial \mathbf{n}} && \text{on } \partial B,\\
\Big(\frac{1}{2} - \mathcal{K}\Big)u + \mathcal{V}\frac{\partial u}{\partial \mathbf{n}} &= 0 && \text{on } \partial B \cup \Gamma.
\end{aligned}
$$



Linear system of equations:

$$
\begin{bmatrix} \mathbf{A}+\mathbf{W} & (\mathbf{K}-\mathbf{B})^T \\ \mathbf{B}-\mathbf{K} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \sigma \end{bmatrix} = \mathbf{h}
$$

- solve saddle point problem by Bramble-Pasciak-CG
- two block preconditioning: BPX + wavelet preconditioning
- synchronous nested iteration:

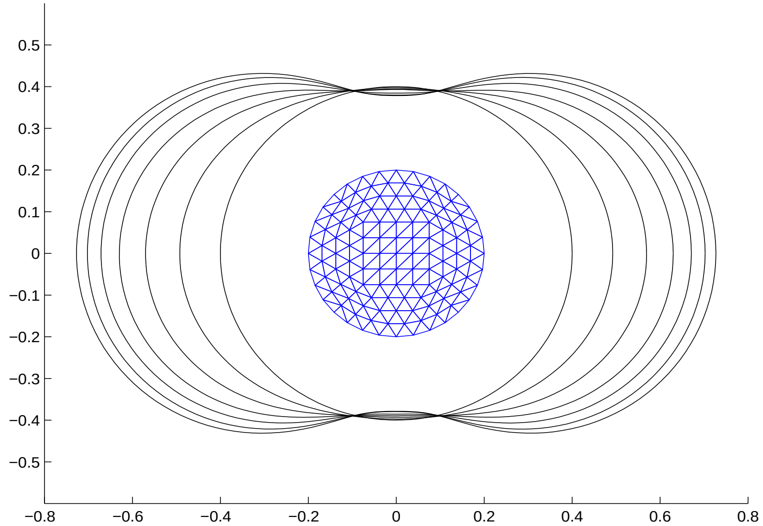
initialization: compute u_0 and p_0
for all $\ell = 1, 2, \ldots, L$ do begin
  compute u_ℓ with initial guess $u_{\ell-1}$
  compute p_ℓ with initial guess $p_{\ell-1}$
end

$L^2(B)$-Tracking type functional

Consider the shape problem $J(\Omega) = \int_B (u - u_d)^2 d\mathbf{x} \to \min$ where

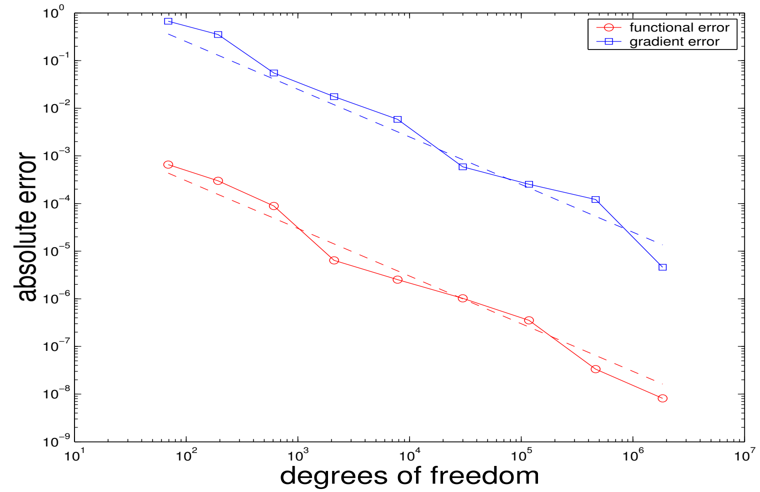
$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma.$$

h_x	h_y	$H^{1/2}([0,2\pi])$-shape error	cpu-time
0.4	0.4	8.4e-5	618
0.5	0.4	3.8e-2	696
0.6	0.4	1.6e-1	706
0.7	0.4	2.9e-1	687
0.8	0.4	5.2e-1	697
0.9	0.4	7.5e-1	680
1.0	0.4	1.0	708



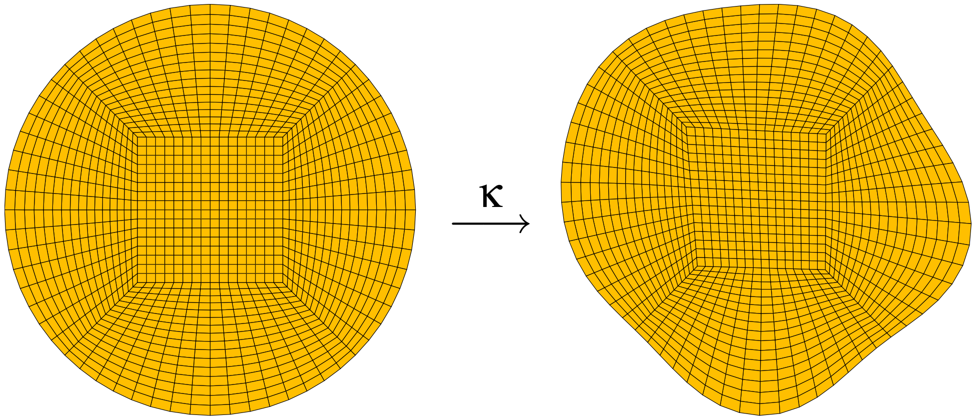
Approximation errors of the functional data:

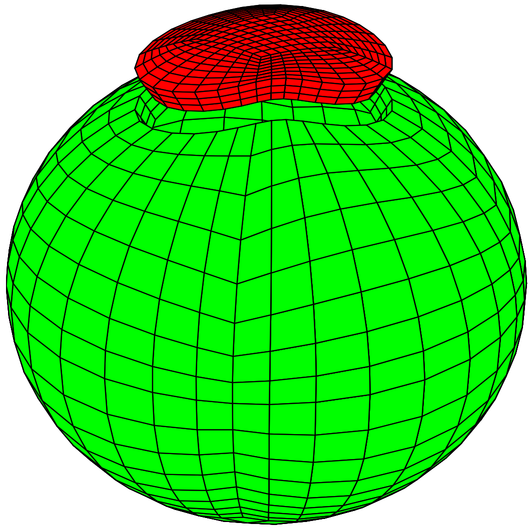
$$|J(\Omega) - J(\Omega)| = O(h^2)$$
$$\|\nabla J(\Omega) - \nabla J(\Omega)\| = O(h)$$



General functionals / state equations

- **Finite element methods**





- **Fictitious domain methods**

  $\longrightarrow$ joint work with Mario Mommer

General shape representations

Consider parametric representation

$$\gamma : \mathbb{S}^2 \to \Gamma, \qquad \gamma(\widehat{\mathbf{x}}) = \begin{bmatrix} \gamma_1(\widehat{\mathbf{x}}) \\ \gamma_2(\widehat{\mathbf{x}}) \\ \gamma_3(\widehat{\mathbf{x}}) \end{bmatrix} = \sum_{n,m} \begin{bmatrix} a_{n,1}^m \\ a_{n,2}^m \\ a_{n,3}^m \end{bmatrix} Y_n^m(\widehat{\mathbf{x}})$$

and boundary variations

$$\gamma_\varepsilon^{(i,n,m)}(\widehat{\mathbf{x}}) = \gamma(\widehat{\mathbf{x}}) + \varepsilon \mathbf{e}_i Y_n^m(\widehat{\mathbf{x}}).$$

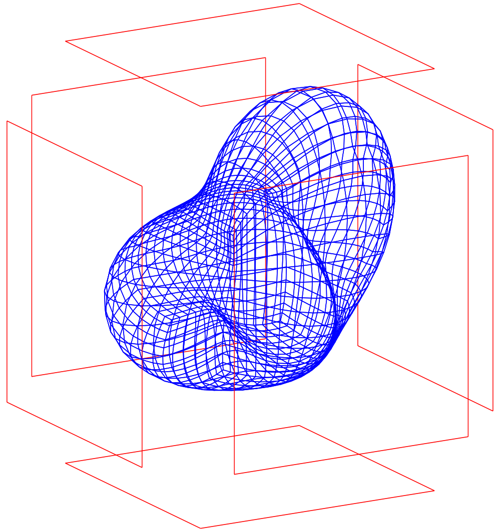
- no unique representation of Γ
- $\gamma_1, \gamma_2, \gamma_3 \in C^{2,\alpha}(\mathbb{S}^2) \not\Rightarrow \gamma \in C^{2,\alpha}(\mathbb{S}^2)$
- mesh might degenerate

Regularization:

$$J(\Omega) + \beta M(\Omega) \to \min, \qquad M(\Omega) := \int_{\mathbb{S}} \left\| \left[\left\langle \frac{\partial \gamma}{\partial \mathbf{t}_i}, \frac{\partial \gamma}{\partial \mathbf{t}_j} \right\rangle \right]_{i,j=1,2} - \frac{|\Gamma|^2}{|\mathbb{S}^2|^2} \cdot \mathbf{I} \right\|_F^2 d\sigma_{\mathbf{x}}$$

Electromagnetic shaping (3D)

Given: bubble $\Omega \subseteq \mathbb{R}^3$ of liquid metal levitating in a magnetic field generated by conductors



Shape problem: (Pierre/Roche)

$$J(\Omega) = A\int_\Gamma 1 d\sigma_{\mathbf{x}} + B\int_\Omega x_3 d\mathbf{x} - \int_{\Omega^c} \|\mathbf{B}\|^2 d\mathbf{x} \to \min,$$

where $\mathbf{B}$ satisfies the magnetostatic Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} && \text{in } \Omega^c, \\ \nabla \mathbf{B} &= 0 && \text{in } \Omega^c, \\ \langle \mathbf{B}, \mathbf{n} \rangle &= 0 && \text{on } \Gamma, \\ \mathbf{B} &= O(\|\mathbf{x}\|^{-2}) && \text{as } \|\mathbf{x}\| \to \infty, \end{aligned}$$

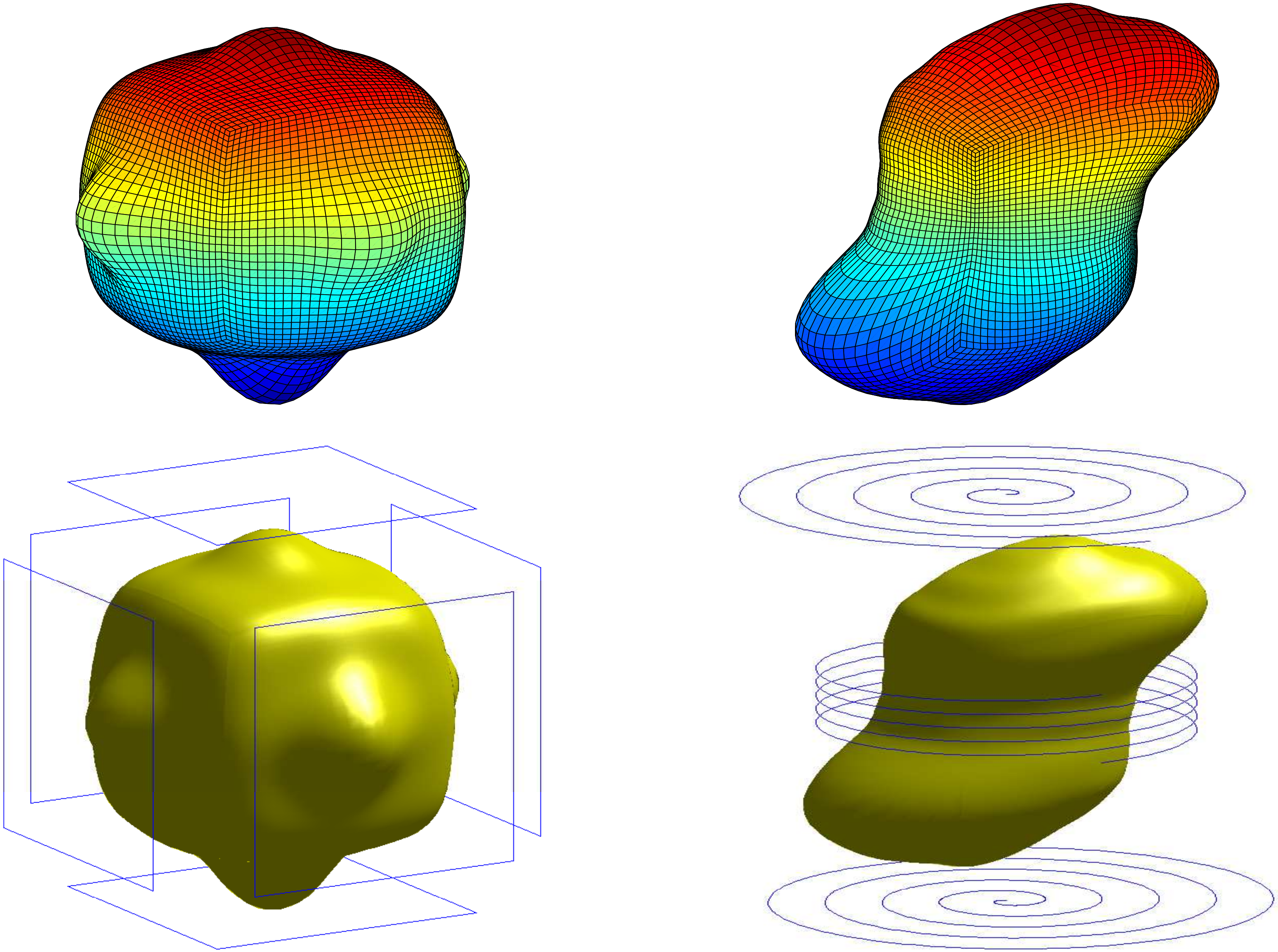
subject to $|\Omega| = V_0$.

Shape gradient:

$$\nabla J(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V}, \mathbf{n} \rangle \left\{ 2A\mathcal{H} + Bx_3 + \|\mathbf{B}\|^2 \right\} d\sigma_{\mathbf{x}}$$

$\rightsquigarrow$ physical equilibrium condition: $2A\mathcal{H} + Bx_3 + \|\mathbf{B}\| = p \equiv const.$ on $\Gamma^\star$

Electromagnetic shaping (3D)



Inverse obstacle scattering **[joint work with Thorsten Hohage]**

Direct problem: Given $\Omega \subset \mathbb{R}^3$ and the incident field $u_i = e^{i\kappa\langle \mathbf{x},\mathbf{d}\rangle}$, find the scattered wave u_s such that for $u := u_i + u_s$

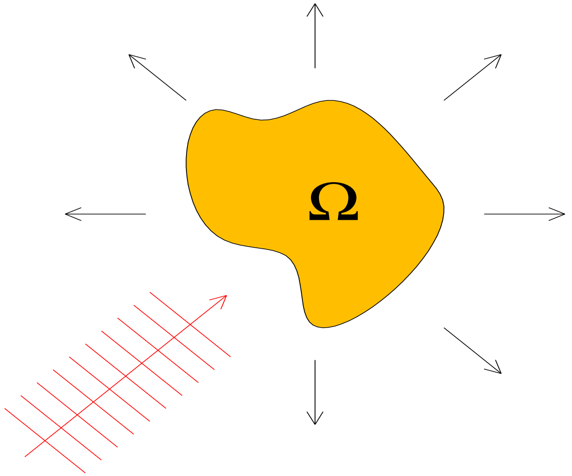
$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \qquad \rightsquigarrow \text{exterior Helmholtz equation}$$

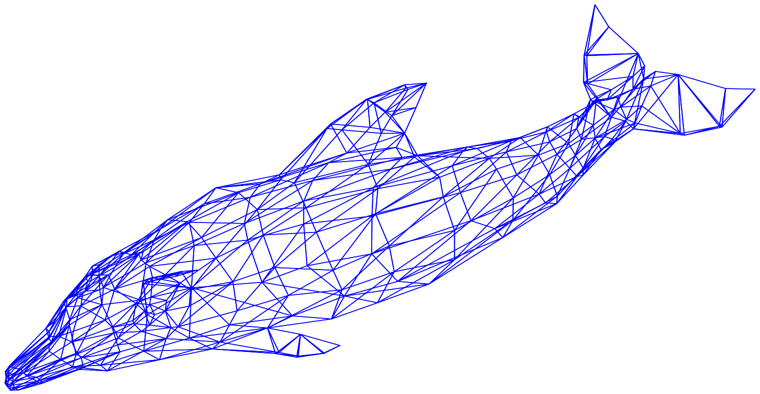
$$u = 0 \quad \text{on } \Gamma, \qquad \rightsquigarrow \text{sound-soft obstacle}$$

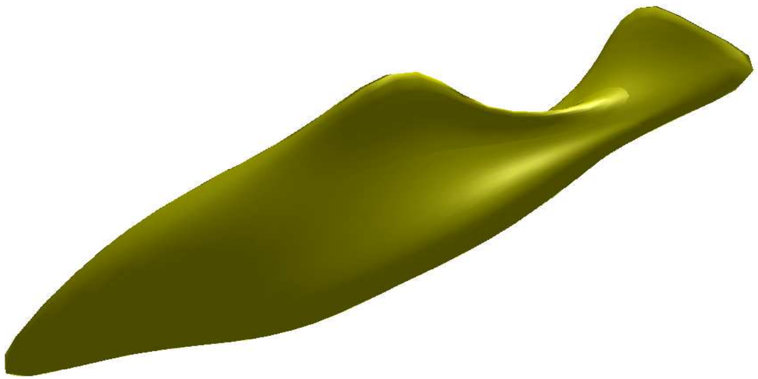
$$\lim_{r\to\infty} r\left\{\frac{\partial u_s}{\partial r} - i\kappa u_s\right\} = 0 \quad r := \|\mathbf{x}\|. \qquad \rightsquigarrow \text{Sommerfeld radiation condition}$$

Inverse problem: Given the direction $\mathbf{d}$ and measurements u_∞^δ of the far field pattern u_∞ of the scattered wave, find the domain Ω.

$$\rightsquigarrow J(\Omega) = \int_{\mathbb{S}} (u_\infty - u_\infty^\delta)^2 d\sigma \to \min$$







Concluding remarks

- we introduced a canonical functional analytical setting for shape optimization problems
- by analysing the shape Hessian we are able to distinguish well-posed and ill-posed shape optimization problems
- we discussed the efficient numerical solution of the state equation by BEM, FEM, and coupling of FEM and BEM
- we presented a technique to approximate general shapes

Thank you for your attention!

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