The range of the non-Abelian X-ray transform

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Joint work with G.P. Paternain

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UNIVERSITÄT BONN

Looking for structure in the range of some two-dimensional inverse problems

similarities and limitations illustrated at some linear and non-linear examples

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Four inverse problems in two-dimensions – what is the range?

(1) Linear X-ray 1-form f on M $\downarrow$ Integrals along geodesics	**(2) Non-Abelian X-ray** Connection A on $M \times \mathbb{C}^n$ $\downarrow$ Parallel transport along geodesics
(3) Calderón problem Riemannian metric g on M $\downarrow$ DN-map of Δ_g	**(4) Scattering problem** Riemannian metric g on M $\downarrow$ Scattering relation of geodesic flow

Four inverse problems in two-dimensions – what is the range?

(1) Linear X-ray	**(2) Non-Abelian X-ray**
1-form f on M $\downarrow$ Integrals along geodesics	Connection A on $M \times \mathbb{C}^n$ $\downarrow$ Parallel transport along geodesics
(3) Calderón problem	**(4) Scattering problem**
Riemannian metric g on M $\downarrow$ DN-map of Δ_g	Riemannian metric g on M $\downarrow$ Scattering relation of geodesic flow

- **Simple setting:** Injectivity of $\{\text{unknown}\} \to \{\text{data}\}$ is understood.
- **Upshot:** for (1),**(2)**,(3) we also understand the range, but (4) is harder
- **New:** B.-Paternain: The transport Oka-Grauert principle for simple surfaces — Journal de l'École Polytechnique, 2023

Setting

Throughout (M, g) is a **simple surface**, that is,

- ∂M is strictly convex;
- all geodesics reach the boundary (non-trapping);
- there are no conjugate points.

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Protagonists:

$$\begin{aligned}
SM &= \{(x,v)\in TM : g(v,v)=1\}\\
\partial_\pm SM &= \{(x,v)\in \partial SM : \pm g(\nu(x),v)\geq 0\}\\
X &= \text{generator of the geodesic flow}\\
&\quad \big(X: C^\infty(SM)\to C^\infty(SM)\big)\\
\alpha &= \text{scattering relation of the geodesic flow}\\
&\quad \big(\alpha\in \mathrm{Diff}(\partial SM)\big)\\
\Omega_k &= \{u\in C^\infty(SM) : u(x,e^{it}v)=e^{ikt}u(x,v)\},\ k\in\mathbb{Z}
\end{aligned}$$

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- **Rk.:** $X:\Omega_k\to\Omega_{k-1}\oplus\Omega_{k+1}$
- **Def.:** Call $w\in C^\infty(SM)$ **fibrewise holomorphic** if $w\in\oplus_{k\geq 0}\Omega_k$.

Linear X-ray – [Pestov–Uhlmann, 2004]

Definition (X-ray transform on 1-forms, valued in $\mathfrak{u}(1) = i\mathbb{R}$)

We define $I_1 : C^\infty(M, T^*M \otimes \mathfrak{u}(1)) \to C^\infty(\partial_+ SM, \mathfrak{u}(1))$ by

$$I_1 f(x, v) = \text{Integral of } f \text{ along geodesic } \gamma_{x,v}$$

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Step 1: Smoothly structure the range

If $f_0, f_1 \in C^\infty(M, T^*M \otimes \mathfrak{u}(1)) \subset \Omega_{-1} \oplus \Omega_1$, then

- solve$^{(\dagger)}$

$$\boxed{Xw = f_0 - f_1} \quad w \in C^\infty(SM)$$

- restrict to ∂SM:

$$\boxed{I_1 f_1 + w \circ \alpha = w + I_1 f_0} \quad w \in C^\infty(\partial SM)$$

(†) Thm.: $X \colon C^\infty(SM) \to C^\infty(SM)$ is onto

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($\dagger$) Thm.: $X\colon \oplus_{k\geq 0}\Omega_{2k} \to \oplus_{k\geq -1}\Omega_{2k+1}$ is onto [Salo-Uhlmann, 2011]

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Step 3: Parametrise the range

- Fix $f_0 = 0$ as anchor, then for any other $f \in C^\infty(M, T^*M \otimes \mathfrak{u}(1))$:

$$\boxed{I_1 f = w - (w \circ \alpha)} \qquad \begin{array}{l} w \in C^\infty(\partial SM) \\ w \text{ fibrewise holomorphic (+even)} \end{array}$$

- Given $h \in C^\infty(\partial SM, \mathbb{R})$ (even), solve Riemann–Hilbert problem:

$$\boxed{h = w + \bar{w}} \quad w \text{ as above} \quad \left(\rightsquigarrow w = \frac{1}{2}(\mathrm{Id} + iH_+)h \right)$$

- Restrict h to $\partial_+ SM$, then

$$\boxed{I_1 f = iPh} \quad P = A_-^* H_+ A_+ = \begin{array}{l} \text{Pestov–Uhlmann} \\ \text{boundary operator} \end{array}$$

Template

Step 1 Smoothly structure the range

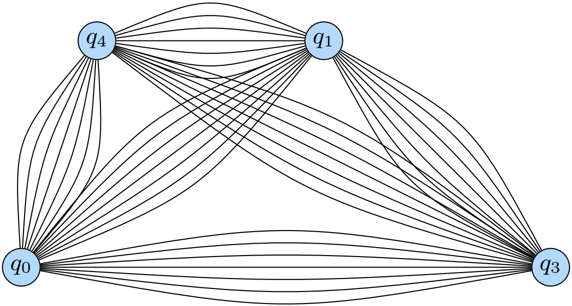
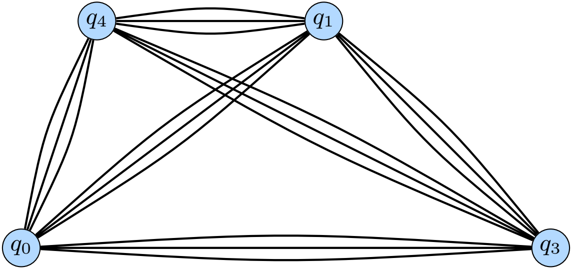


Figure: **Vertices:** Points in the range
Edges: Conjugations between them

Template

Step 1 Smoothly structure the range

Step 2 Holomorphically structure the range



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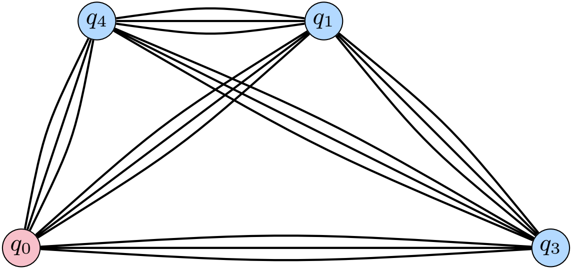

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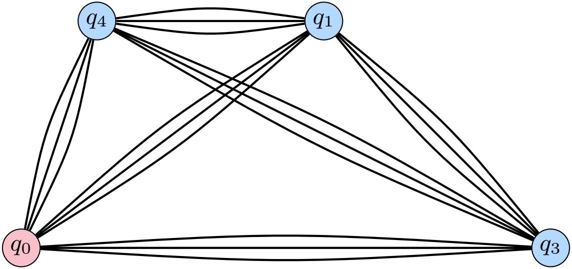

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- Let's see how far we get with the other problems!

Non-Abelian X-ray – [B.–Paternain, 2023]

Definition (Non-Abelian X-ray transform on unitary connections)

We define $C\colon C^\infty(M, T^*M \otimes \mathfrak{u}(n)) \to C^\infty(\partial_+ SM, U(n))$ by

$$C_A(x,v) = \text{Parallel transport of connection } d + A \text{ along } \gamma_{x,v}$$

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Step 1: Smoothly structure the range

If $A_0, A_1 \in C^\infty(M, T^*M \otimes \mathfrak{u}(n))$, then

- solve$^{(\dagger)}$

$$\boxed{W^{-1}(X + A_1)W = A_0} \quad W \in C^\infty(SM, GL(n, \mathbb{C}))$$

- restrict to ∂SM:

$$\boxed{C_{A_1}(W \circ \alpha) = W C_{A_0}} \quad W \in C^\infty(SM, GL(n, \mathbb{C}))$$

($\dagger$) Thm.: $C^\infty(SM, GL(n, \mathbb{C}))$ acts transitively $C^\infty(SM, \mathfrak{gl}(n, \mathbb{C}))$

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(†) Thm.: $\mathbb{G} = \{W \text{ as above}\}$ acts transitively on $\oplus_{k\geq -1}\Omega_{2k+1} \otimes \mathfrak{gl}(n,\mathbb{C})$ [Transport Oka-Grauert Principle]

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Step 3: Parametrise the range

- Fix $A_0 = 0$ as anchor, then for any other $A \in C^\infty(M, T^*M \otimes \mathfrak{u}(n))$:

$$\boxed{C_A = W(W^{-1} \circ \alpha)} \qquad \begin{array}{l} W \in C^\infty(SM, GL(n,\mathbb{C})) \\ W, W^{-1} \text{ fibrewise holomorphic } (+\text{even}) \end{array}$$

- Given $H \in C^\infty(\partial SM, \mathrm{Her}_n^+)$, solve the Riemann–Hilbert problem

$$\boxed{H = W^* W} \quad W \text{ as above} \quad \Big(\rightsquigarrow \text{Birkhoff theorem} \Big)$$

- Restrict H to $\partial_+ SM$, then:

$$\boxed{C_A \equiv \mathcal{P}(H) \mod C^\infty_{\mathrm{Id}}(\partial M, U(n))} \quad \mathcal{P} = \begin{array}{l} \text{nonlinear Pestov–Uhlmann} \\ \text{boundary operator} \end{array}$$

Definition (DN-map)

We define $\Lambda\colon \mathrm{Riem}(M) \to \mathcal{L}(C^\infty(\partial M))$ by

$$\Lambda_g f = \partial_\nu u \qquad \text{where } \begin{cases} \Delta_g u = 0 & M \\ u = f & \partial M. \end{cases}$$

Calderón problem – [Sharafutdinov, 2011]

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Step 1: Smoothly structure the range

If $g_0, g_1 \in \mathrm{Riem}(M)$, then

- Solve[†]

$$\boxed{\varphi^* g_0 = e^{2\sigma} g_1} \quad (\varphi, \sigma) \in \mathrm{Diff}(M) \times C^\infty(M, \mathbb{R})$$

- Restrict to ∂M:

$$\boxed{\Lambda_{g_1} \varphi^* = \varphi^* \Lambda_{g_0}} \quad \varphi \in \mathrm{Diff}_{\mathrm{Id}}(\partial M)$$

(†) Riemann mapping theorem

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(†) Riemann mapping theorem

- As good as it gets (?)

Scattering problem

Definition (Scattering data)

To a simple metric g we associate the scattering data $\alpha_g \in \mathrm{Diff}(\partial SM_g)$ by

$$\begin{aligned} \alpha_g(x,v) &= (\gamma_{x,v}(\tau), \dot{\gamma}_{x,v}(\tau)), \quad (x,v) \in \partial_+ SM_g \\ \alpha_g \circ \alpha_g &= \mathrm{Id} \end{aligned}$$

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Step 1: Smoothly structure the range

For two simple metrics g_0 and g_1

- solve(†)

$$\boxed{\phi_* X_{g_0} = a X_{g_1}} \quad (\phi, a) \in \mathrm{Diff}(SM_{g_0}, SM_{g_1}) \times C^\infty(SM_{g_1})$$

- restrict to ∂SM:

$$\boxed{\alpha_{g_0} \circ \phi = \phi \circ \alpha_{g_1}} \quad \phi \in \mathrm{Diff}(\partial SM_{g_0}, \partial SM_{g_1})$$

(†) **Thm.:** The geodesic flows of any two simple metrics are orbit conjugate.

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Step 2: Holomorphically structure the range

- Is there a natural notion of *fibrewise holomorphicity* for diffeomorphisms $\phi\colon SM_{g_0} \to SM_{g_1}$? — `Yes`
- Can the whole range be reached by conjugation with these? — `No`$^{(\dagger)}$

($\dagger$) Intimately connected to the complex geometry of transport twistor space $\rightsquigarrow$ ongoing work with F. Monard and G.P. Paternain