The homology of Moduli Spaces of Riemann Surfaces

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Moduli spaces and mapping class groups

Introduction A motivating question would be the following: How can one classify the complex structures on a two dimensional manifold F? The first huge step towards a satisfactory answer, is the construction of the moduli space $\mathfrak{M}$. Its underlying points are in one-to-one correspondence with the set of equivalence classes of complex structures. The study of these moduli spaces relates topology, geometry, algebra and mathematical physics.

The moduli space $\mathfrak{M}^m_{g,n}$ Fix $g \geq 0$, $m \geq 0$ and $n \geq 1$. Our data for a surface consists of

(1) a Riemann surface F of genus g;

(2) a set $\mathcal{P} = \{P_1, \ldots, P_m\} \subset F$ of m distinct points;

(3) non-vanishing tangential directions $\mathcal{X} = (X_1, \ldots, X_n)$ at points $\mathcal{Q} = (Q_1, \ldots, Q_n)$ disjoint from $\mathcal{P}$.

Two surfaces $[F, \mathcal{P}, \mathcal{Q}, \mathcal{X}]$ and $[F', \mathcal{P}', \mathcal{Q}', \mathcal{X}']$ are equivalent if and only if there is a bihomolorphic map $\varphi\colon F \longrightarrow F'$ respecting the structure. The set of equivalence classes embody the moduli space of Riemann surfaces $\mathfrak{M}^m_{g,n}$. The condition $n \geq 1$ ensures that it is both a manifold of dimension $6g - 6 + 2m + 4n$ and a classifying space $B\Gamma^m_{g,n}$ for the mapping class group (because the action of $\Gamma^m_{g,n}$ on the Teichmüller space is well behaved).

The mapping class group $\Gamma^m_{g,n}$ Let F be smooth, oriented, of genus g with $\mathcal{P}$, $\mathcal{X}$ and $\mathcal{Q}$ as above. Let

$$Diff^+ = Diff^+(F, \mathcal{P}, \mathcal{Q}, \mathcal{X}) = \{\varphi\colon F \xrightarrow{\cong} F \mid \text{smooth, orientation preserving, respecting } \mathcal{P},\ \mathcal{X} \text{ and } \mathcal{Q}\}\,.$$

with the C^∞-Whitney topology and let $Diff^+_0 \subset Diff^+$ be the subspace of diffeomorphisms isotopic to the identity. The usual composition of maps turns $Diff^+$ into a topological group with $Diff^+_0$ a contractible subgroup. The mapping class group is

$$\Gamma^m_{g,n} = Diff^+(F, \mathcal{P}; \mathcal{Q}, \mathcal{X}) \;/\; Diff^+_0(F, \mathcal{P}; \mathcal{Q}, \mathcal{X}) = \pi_0 Diff^+(F, \mathcal{P}; \mathcal{Q}, \mathcal{X})\,.$$

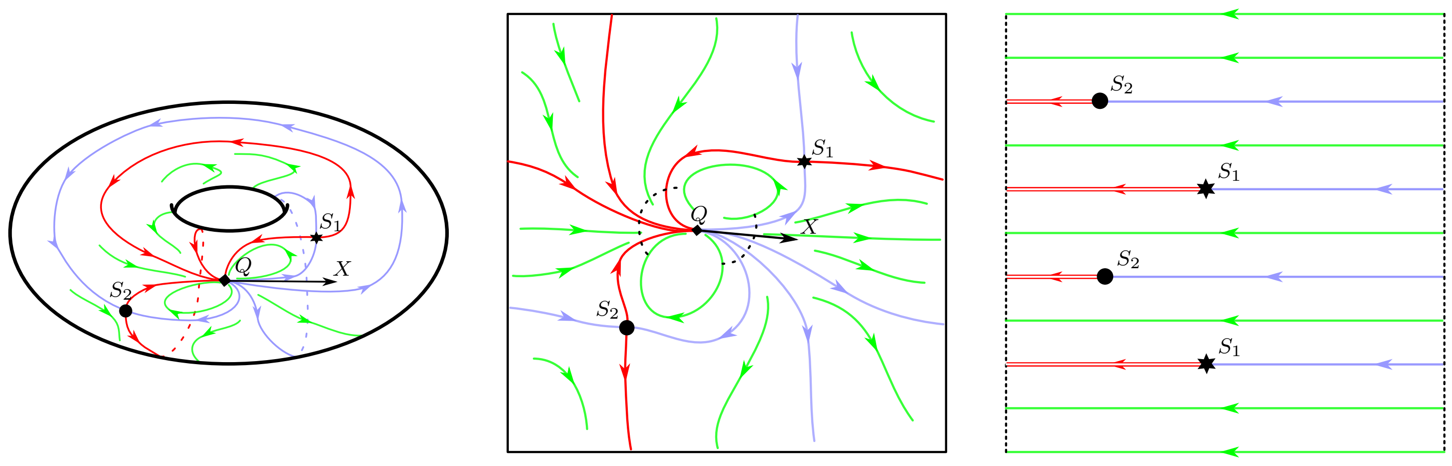
Instead of fixing directions $\mathcal{X}$ at $\mathcal{Q}$, we remove an open small disc around every Q_i and obtain a compact surface $\hat{F}$ with n boundary circles which are required to be fixed in a small ε-neighbourhood. This gives isomorphic groups; both are finitely presented by Dehn twists.

$$\Gamma^m_{g,n} = Diff^+(\hat{F}, \mathcal{P}; \partial\hat{F}) \;/\; Diff^+_0(\hat{F}, \mathcal{P}; \partial\hat{F}) = \pi_0 Diff^+(\hat{F}, \mathcal{P}; \partial\hat{F})\,.$$

Hilbert uniformization A method providing a comfortable model for $\mathfrak{M}^m_{g,n}$ is introduced in [Böd1]. In order to ease the discussion of the uniformization process, we provide a pictorial example on the next page, where $g = 1$, $m = 0$ and $n = 1$. Given a complex surface $[F] \in \mathfrak{M}^m_{g,n}$ we choose a map $u\colon F \longrightarrow \hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ which is harmonic away from $\mathcal{P}$ and $\mathcal{Q}$. Moreover, we assert a dipole at every $Q_i \in \mathcal{Q}$ in direction X_i and with a logarithmic sink at every $P_j \in \mathcal{P}$. The flow of steepest descent has finitely many critical points $S_1, \ldots, S_k$. The union of $\mathcal{Q}$, $\mathcal{P}$, all the S_l and the flow lines leaving the S_l constitute the critical graph K drawn in red.

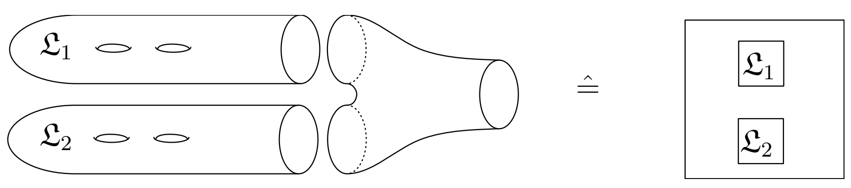
Observe that $F - K$ consist of exactly n contractible components because every flow line starts near exactly one Q_i. The process of "straightening the remaining flow lines" defines a bihomolorphic map $u + iv$ from $F - K$ into the complex plane. The image is $\mathbb{C}$ minus a finite number of horizontal half-rays running to the left; this we call a slit configuration.

Additive structures and the harmonic compactification



The space of such maps $u + iv$ is denoted by $\mathcal{H}^m_{g,n}$. It is a bundle $\mathcal{H}^m_{g,n} \xrightarrow{\simeq} \mathfrak{M}^m_{g,n}$ and the choices we made constitute the fibre which is contractible. The space $\mathcal{H}^m_{g,n}$ is homeomorphic to the space of admissible slit configurations denoted by $\mathfrak{Par}^m_{g,n}$. We remark that a similar procedure results in another model for $\Gamma^m_{g,n}$, namely in the space $\mathfrak{Rad}^m_{g,n}$ of admissible slit configurations on n annuli.

The E_2-space structure The data of a slit picture $\mathfrak{L} \in \mathfrak{Par}^m_{g,1}$ consists of the endpoints of the half-rays and certain glueing information. Thus, $\mathfrak{L}$ is inscribed in a square of finite area. Placing two slit pictures into disjoint squares in $\mathbb{C}$ defines an H-space structure on $\mathfrak{Par} = \coprod^m_{g,1} \mathfrak{Par}^m_{g,1}$. On $\mathfrak{M} = \coprod_{g,m} \mathfrak{M}^m_{g,1}$, the corresponding operation is induced by joining the surfaces by a pair of pants.



More generally, the little 2-cubes operad $\tilde{C}(\mathbb{C}) = \coprod_{k \geq 0}\{k \text{ disjoint, paraxial squares in } \mathbb{C}\}$ acts on $\mathfrak{Par}$. Consequently, $H_*(\mathfrak{Par}) \cong H_*(\mathfrak{M})$ is not only a commutative Pontryargin ring, but a Dyer–Lashof algebra. We discuss its structure in a moment.

The harmonic compactification The space of radial slit configurations $\mathfrak{Rad}^m_{g,1}$ is a model for the moduli space of Riemann surfaces $\mathfrak{M}^m_{g,1}$. It is not compact; but allowing certain degenerations of handles and boundary curves, we obtain the harmonic compactification $\mathfrak{M}^m_{g,1} \subset \overline{\mathfrak{M}}^m_{g,1}$. In [EK], it is identified with a space of Sullivan diagrams which is used used in [Wah] to classify all natural operations on the Hochschild complex of symmetric Frobenius algebras. Besides computations for small g and m, we have the following result.

Theorem (B.–Egas 2016$^+$). *Given parameters $g \geq 0$ and $m \geq 1$, the space of Sullivan diagrams $\mathscr{SD}^m_{g,1}$ is highly connected i.e.*

$$\pi_*(\mathscr{SD}^m_{g,1}) = 0 \quad \textit{for} \quad * \leq m - 2\,.$$

The (un)stable situation

Results in the so called stable range The Harer stabilization theorem states, that the multiplication with the generator in $H_0(\Gamma^0_{1,1})$ induces an isomorphism $H_*(\Gamma^0_{g,1}) \xrightarrow{\cong} H_*(\Gamma^0_{g+1,1})$ if $* \leq \frac{2}{3}g - 1$. Thus $\Gamma_{\infty,1} = \cup_g \Gamma_{g,1}$ is an approximation of every $\Gamma_{g,1}$ in this so called stable range. In [MW] Madsen and Weiss construct a Thom spectrum $MT(d)^+$ detecting the homotopy type of a cobordism category. As a special case, a group completion theorem yields a homology isomorphism $\mathbb{Z} \times B\Gamma_{\infty,1} \longrightarrow \Omega^\infty MT(2)^+$. This proves a conjecture by Mumford.

Theorem (Madsen–Weiss 2002). *The rational cohomology of $\Gamma_{\infty,1}$ is*

$$H^*(\Gamma_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$$

with κ_i the Mumford–Morita–Miller characteristic classes for surface bundles. In particular, $H_*(\Gamma^0_{g,1}; \mathbb{Q})$ is known in the stable range $* \leq \frac{2}{3}g - 1$.

Using a different technique, the stabilization results carry over to the harmonic compactification.

Theorem (B.–Egas 2016$^+$). *Let $g \geq 0$ and $m \neq 2$. The stabilization map $\mathscr{SD}^m_{g,1} \longrightarrow \mathscr{SD}^m_{g+1,1}$ is highly connected i.e.*

$$\pi_*(\mathscr{SD}^m_{g,1}) \xrightarrow{\cong} \pi_*(\mathscr{SD}^m_{g+1,1}) \quad \textit{for} \quad * \leq g + m - 2\,.$$

Moreover, we construct infinite families of non-trivial homology classes. However, indentifying the stable compactification $\mathscr{SD}^m_{\infty,1}$ or its (rational) homology is a difficult task.

Computations in the unstable range The space of parallel slit domains $\mathfrak{Par}^m_{g,n}$ is a combinatorial, relative manifold, i.e. $\mathfrak{Par}^m_{g,n} \cong \mathbb{P} - \mathbb{P}'$ with $(\mathbb{P}, \mathbb{P}')$ a pair of compact cell complexes. The homology of $\mathfrak{M}^m_{g,n}$ is therefore Poincaré dual to the cohomology of $\mathbb{P}/\mathbb{P}'$. Computations for $2g + m < 6$ were done by Ehrenfried, Mehner and Wang using this model; and Godin using another model. Bödigheimer introduces an elegant filtration on $\mathbb{P}$ in [Böd2]. We state some of our results for $2g + m = 6$ using this filtration.

Theorem (Bödigheimer, B., Hermann 2014). *The rational betti numbers of the moduli spaces are as follows.*

	$*=0$	$*=1$	$*=2$	$*=3$	$*=4$	$*=5$	$*=6$	$*=7$	$*=8$	$*=9$
$\dim_\mathbb{Q} H_*(\mathfrak{M}^4_{1,1})$	1	1	0	2	3	2	1	0	0	0
$\dim_\mathbb{Q} H_*(\mathfrak{M}^2_{2,1})$	1	0	1	3	0	2	2	0	0	0
$\dim_\mathbb{Q} H_*(\mathfrak{M}^0_{3,1})$	1	0	1	1	0	1	1	0	0	1

Unstable homology via homology operations The unit tangent bundle of the universal surface bundle is the fibre of the forgetful map $\mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^m_{g,0}$. Using this fibration, we detect an infinite family of non-vanishing, rational homology classes for varying g. These classes perish in the stabilization process. Moreover, Bödigheimer and the author provide some relations between generators via operadic homology operations.

Unstable homology via braid groups I The moduli space $\mathfrak{M}^m_{0,1}$ is the space of m undistinguishable particles in the plane. Thus, $Br_m = \pi_1(\mathfrak{M}^m_{0,1}) = \Gamma^m_{0,1}$ is the braid group on m stands. Using the theory of iterated loop spaces, Cohen provides the p-torsion of the integral homology and its description as Dyer–Lashof algebra. The classical result by Arnold and Fuks is then obtained as a corollary.

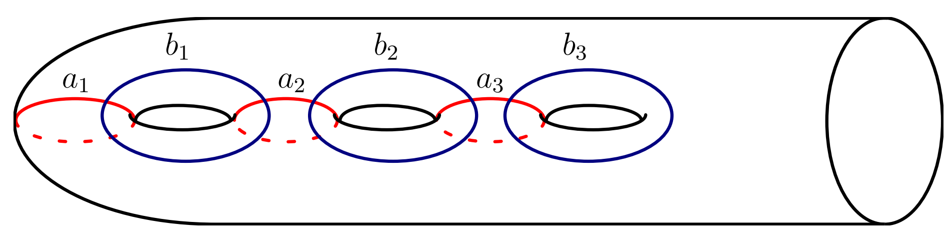
Forgetting the marked points defines a fibration $\mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^0_{g,1}$ with fibre $C^m(F_{g,1})$, the unordered configuration space on the closed surface. Adding a marked point near the boundary curve, defines a map $\mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^{m+1}_{g,1}$ that is compatible with projektion to $\mathfrak{M}^0_{g,1}$. The induced map in homology $H_0(\mathfrak{M}^1_{0,1}; \mathbb{Z}) \otimes H_*(\mathfrak{M}^m_{g,1}; \mathbb{Z}) \longrightarrow H_*(\mathfrak{M}^{m+1}_{g,1}; \mathbb{Z})$ is the multiplication with the generator in $H_0(\mathfrak{M}^1_{0,1})$. It is split-injective by [BT1].

On the unstable homology

Using the braid group on two strands, we obtain infinite families of non-trivial (unstable) homology classes.

Theorem (B. 2015$^+$). *The generator $b \in H_1(Br_2; \mathbb{F}_2) = H_1(\mathfrak{M}^2_{0,1}; \mathbb{F}_2)$ spans a polynomial ring $\mathbb{F}_2[b]$ inside $H_*(\mathfrak{M}; \mathbb{F}_2)$. Regarding $H_*(\mathfrak{M}; \mathbb{F}_2)$ as a module over $\mathbb{F}_2[b]$, it is torsion free.*

Unstable classes via braid groups II In the last paragraph, we identified the k^{th} braid group Br_k with $\Gamma^k_{0,1}$. Sending the braid generators σ_i to certain Dehn twists, [BT2] construct more families of maps from Br_k to $\Gamma^m_{g,1}$. Let us review one of these. The map $\phi_g\colon Br_{2g} \longrightarrow \Gamma^0_{g,1}$ sends the generators $\sigma_1, \ldots, \sigma_{2g-1}$ to the Dehn twists along the simple closed curves $a_1, b_1, \ldots, a_g, b_g$ drawn red and blue in picture below.



The stable version $\phi_\infty\colon Br_\infty \longrightarrow \Gamma^0_{\infty,1}$ comes from a map of double-loop spaces that is null-homotopic [BT2]. Therefore, ϕ_g is the trivial map in homology in the stable range. The same is true for most maps constructed in [BT2]. However, it turns out that some of these are non-trivial in the unstable range.

Proposition (B. 2016$^+$). *For $g \leq 2$, the map ϕ_g induces a split injection in homology outside the stable range. Moreover, we have a canonical map $\psi_2\colon Br_6 \longrightarrow \Gamma^0_{2,1}$, inducing a split injection $\mathbb{Z}/3\mathbb{Z} \cong H_4(Br_6; \mathbb{Z}) \longrightarrow H_4(\Gamma^0_{2,1}; \mathbb{Z})$.*

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