On a vector bundle which cuts, bounces, embeds and measures waists

Pavle Blagojević

Freie Universität Berlin & Matematički Institut SANU Belgrade

joint work with Frederick R. Cohen, Michael Crabb, Wolfgang Lück & Günter M. Ziegler

The vector bundles associated with the natural permutation representation $U_n := \mathbb{R}^n$, and the standard representation $W_n := \{(y_1, \ldots, y_n) \in U_n : \sum y_i = 0\}$, of the symmetric group $\mathfrak{S}_n$ over a free S-spaces X, $S \subseteq \mathfrak{S}_n$, are defined by

$$\xi = \xi_{(\mathfrak{S}_n, S, X)} : \qquad U_n \longrightarrow X \times_S U_n \longrightarrow X/S,$$

$$\zeta = \zeta_{(\mathfrak{S}_n, S, X)} : \qquad W_n \longrightarrow X \times_S W_n \longrightarrow X/S.$$

These bundles, in the case when $S = \mathfrak{S}_n$ and $X = \mathrm{F}(M, n)$ is the classical configuration space of n pairwise distinct points on M, were originally studied and very efficiently used by Cohen, Cohen & Handel, Chisholm, Cohen, Mahowald & Milgram, Bödigheimer, Cohen & Taylor and many others. In the last decade, problems of

— the existence of convex measure partitions $\big($the twisted Euler class of $\zeta^{\oplus(d-1)}_{(\mathfrak{S}_n, \mathfrak{S}_n, \mathrm{F}(\mathbb{R}^d, n))}\big)$,

— the existence of ℓ-skew embeddings $\big($the dual Steifel–Whitney classes of $\xi^{\oplus(d+1)}_{(\mathfrak{S}_n, \mathfrak{S}_n, \mathrm{F}(\mathbb{R}^d, n))}\big)$,

— an estimation of waists of manifolds $\big($the top Steifel–Whitney class of $\zeta^{\oplus(d-1)}_{(\mathfrak{S}_{2^m}, \mathfrak{S}^{(2)}_{2^m}, (S^{d-1})^{2^m-1})}\big)$,

— counting periodic billiard trajectories $\big($any non-zero characteristic class of $\zeta_{(\mathfrak{S}_p, \mathbb{Z}/p, \mathrm{G}(\mathbb{R}^d, n))}\big)$,

motivated Gromov, Ghomi & Tabachnikov, Karasev, Hubard & Aronov, Crabb, and Blagojević, Lück & Ziegler, to start a new study on the properties of these vector bundles.

In this talk we go a bit deeper. Using an embedding of the product of spheres $(S^{d-1})^{2^m-1}$ into the configuration space $\mathrm{F}(\mathbb{R}^d, 2^m)$, which is equivariant, only with respect to the action of a Sylow 2-subgroup $\mathcal{S}_{2^m}$ of $\mathfrak{S}_{2^m}$, we first show that the cohomology ring $H^*(\mathrm{F}(\mathbb{R}^d, 2^m)/\mathfrak{S}_{2^m}; \mathbb{F}_2)$ of the unordered configuration space can be seen as a subring of the cohomology ring

$$H^*((S^{d-1})^{2^m-1}/\mathcal{S}_{2^m}; \mathbb{F}_2) \cong \mathbb{F}_2[V_{m,1}, \ldots, V_{m,m}]/\langle V^d_{m,1}, \ldots, V^d_{m,m}\rangle \oplus \mathrm{I}^*(\mathbb{R}^d, 2^m),$$

where $\mathrm{I}^*(\mathbb{R}^d, 2^m)$ is an ideal, and $\deg(V_{m,r}) = 2^{r-1}$, $1 \le r \le m$.

In the next step we express the Stiefel–Whitney classes of the vector bundles $\xi := \xi_{(\mathfrak{S}_{2^m}, \mathfrak{S}_{2^m}, \mathrm{F}(\mathbb{R}^d, n))}$ in the language of $\mathrm{GL}_m(\mathbb{F}_2)$-invariant Dickson polynomials. Then, using the subgroup $\mathrm{U}_m(\mathbb{F}_2)$ of $\mathrm{GL}_m(\mathbb{F}_2)$, of upper triangular matrices with ones on the main diagonal, we realize the generators $V_{m,r}$ as $\mathrm{U}_m(\mathbb{F}_2)$-invariants. Expressing recursively $\mathrm{GL}_m(\mathbb{F}_2)$-invariants in terms of $\mathrm{U}_m(\mathbb{F}_2)$-invariants we explicitly identify the sequence $w_{2^m-2^0}(\xi), w_{2^m-2^1}(\xi), \ldots, w_{2^m-2^{m-1}}(\xi)$ of Stiefel–Whitney classes in the polynomial part of the cohomology ring $H^*((S^{d-1})^{2^m-1}/\mathcal{S}_{2^m}; \mathbb{F}_2)$.

In this way, we made a step closer towards understanding the ideal generated by the Stiefel-Whitney classes of the vector bundle $\xi := \xi_{(\mathfrak{S}_n, \mathfrak{S}_n, \mathrm{F}(\mathbb{R}^d, n))}$:

$$(w_1(\xi), w_2(\xi), \ldots, w_{n-1}(\xi)) \in H^*(\mathrm{F}(\mathbb{R}^d, n)/\mathfrak{S}_n; \mathbb{F}_2),$$

hoping to give complete answers to some of the questions listed above.