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This is the last set of exercises and will not give points towards qualifying for the oral exam.

Exercise 1. Let $\mathcal{D}$ be the set of dyadic cubes in $\mathbb{R}^d$. Define

$$M_d f(x) = \sup_{Q\in\mathcal{D}} \frac{1_Q(x)}{|Q|} \int_Q |f(y)|\,dy, \qquad M^\# f(x) = \sup_{Q\in\mathcal{D}} \frac{1_Q(x)}{|Q|} \int_Q |f(y) - f_Q|\,dy,$$

where $f_Q = |Q|^{-1} \int_Q |f|$. Prove the good-lambda inequality: For any $\lambda, \epsilon > 0$, it holds

$$|\{M_d f > 2\lambda,\ M^\# f \le \epsilon\lambda\}| \le 2^d \epsilon |\{M_d f > \lambda\}|.$$

(Hint: consider Calderón–Zygmund cubes at level λ. Add and subtract suitable f_Q inside the dyadic maximal function and use the hypothesis as well as properties of the maximal function.)

Exercise 2. (a) Show that for all $p \in (1,\infty)$ there is C depending only on d and p such that

$$\|M_d f\|_p \le C \|M^\# f\|_p \le 2C \|M_d f\|_p.$$

(b) Prove that if $\theta \in (0,1)$ and $p, q > 1$ satisfy $1/p = (1-\theta)/q$, then

$$\|f\|_p \le C \|f\|_q^{1-\theta} \|f\|_{BMO}^{\theta}$$

where C only depends on n, q and p.