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Submit the solutions in groups of two at the lecture on Tuesday, 2018-05-15

Exercise 1. Let $1 \leq p < \infty$. Suppose that there exists a constant $C > 0$ such that for all Schwartz functions f with $\|f\|_p = 1$ one has

$$|\{x : |H(f)(x)| > 1\}| \leq C.$$

Using only this inequality, prove that H maps $L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})$. (Hint: consider dilations)

Exercise 2.

(a) Denote $\varphi_\lambda(x) = \varphi(\lambda x)$. A distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is said to be *homogeneous of degree* α if $f(\varphi_\lambda) = \lambda^\alpha f(\varphi)$ for all test functions φ. Show that if f is homogeneous of degree α, then $\widehat{f}$ is homogeneous of degree $-n - \alpha$.

(b) Let $A \in \mathbb{R}^{d\times d}$ be a rotation matrix. Show that if $\varphi(Ax) = \varphi(x)$, then also $\widehat{\varphi}(A\xi) = \widehat{\varphi}(\xi)$. Show that if the tempered distribution f is invariant under rotations, so is $\widehat{f}$.

(c) Define the Riesz kernel of order $\alpha \in (0, n)$ to be $K_\alpha(x) = |x|^{-n+\alpha}$. Compute its Fourier transform up to a constant.

(d) Prove that

$$\|\partial_\alpha(K_k * f)\|_p \leq C_{k,\alpha,d,p}\|f\|_p$$

for $p \in (1, \infty)$ and $\alpha \in \mathbb{N}^d$ with $\sum_j \alpha_j = k$.