S4A1: GRADUATE SEMINAR ON ALGEBRAIC GEOMETRY: THE WEIL CONJECTURES

This seminar is a continuation of the course on étale cohomology taught in Winter 2016/2017. The Weil conjectures are a series of statements about the generating functions obtained by counting the number of points of smooth projective varieties over finite fields; these generating functions are called zeta functions by analogy with the Riemann zeta function.

Let $\mathbb{F}_q$ be a finite field of characteristic p and let $X/\mathbb{F}_q$ be a scheme of finite type of dimension n. Let $|X|$ denote the set of closed points of X. We can define the zeta function of X as

$$\zeta(X,s) := \prod_{x\in|X|}(1-\#k(x)^{-s})^{-1},$$

which can be rewritten as

$$\zeta(X,s) = \exp\left(\prod_{n=1}^{\infty}\frac{\#x(\mathbb{F}_{q^n})}{n}q^{-ns}\right).$$

This is a function of the complex variable s; the product converges for $\mathrm{Re}(s)$ sufficiently large. For example, the zeta function of $X=\mathbb{P}^n$ is

$$\zeta(X,s) = \frac{1}{(1-q^{-s})(1-q^{1-s})\dots(1-q^{n-s})}$$

Now assume that X is smooth and projective. The Weil Conjectures state the following.

(1) (Rationality) The function $\zeta(X,s)$ is a rational function of q^{-s}.

(2) (Functional equation) There exists an integer N such that

$$\zeta(X,n-s) = \pm q^{N(n/2-s)}\zeta(X,s)$$

(3) (Riemann hypothesis) We can write

$$\zeta(X,s) = \frac{P_1(q^{-s})\dots P_{2n-1}(q^{-s})}{P_0(q^{-s})\dots P_{2n}(q^{-s})},$$

where $P_0(T)=1-T$, $P_{2n}(T)=1-q^nT$ and each P_i is a polynomial with integer coefficients and with roots of absolute value $q^{-i/2}$.

(4) (Relation to Betti cohomology) If R is a finitely generated $\mathbb{Z}$-algebra, with $R \twoheadrightarrow \mathbb{F}_q$ and $R \hookrightarrow \mathbb{C}$ and $\mathcal{X}/\mathrm{Spec}\ R$ is smooth and proper such that $X=\mathcal{X}_{\mathbb{F}_q}$, then the degree of each polynomial P_i matches the ith Betti number of the complex manifold $\mathcal{X}(\mathbb{C})$.

The goal of the seminar is to give a proof of the Weil Conjectures using étale cohomology following [1, 2]. (Additional references are below.) During the first part of the seminar, we will continue to build up some preliminaries from étale cohomology: we will review proper base change and then discuss smooth base change, Poincaré duality and the Grothendieck-Lefschetz trace formula. This will lead us to reinterpret the zeta functions of algebraic varieties in terms of étale cohomology. The properties of étale cohomology will lead directly to the proof of three of the four conjectures (rationality, functional equation, and relationship to Betti cohomology). In the second part of the seminar, we will prove the Riemann hypothesis following [2]. One of the key inputs is the theory of Lefschetz pencils, for which we will also use the reference [5].

Talks

Talk 1: Smooth base change. May 1 2017. Speaker: Ana C.
Discuss locally acyclic morphisms, prove that smooth morphisms are locally acyclic and give applications, including the smooth base change theorem. Reference: Exposé I, 5 of [1].

Talk 2: Poincaré duality. May 15 2017. Speaker: Benjamin L.
Give the motivation from Betti cohomology, discuss the case of curves, define the trace morphism in general and show that it induces a perfect pairing. Reference: Exposé I, 6 of [1]

Talk 3: The Grothendieck-Lefschetz trace formula. May 22 2017. Speaker: Xuanyu P.
Introduce l-adic sheaves, where l is some prime with $l \neq p$. State and prove the Lefschetz trace formula for the Frobenius endomorphism. Reference: Exposé II of [1]

Talk 4: The Weil Conjectures, except the Riemann hypothesis. May 29 2017. Speaker: Mafalda S.
Recall the cohomological interpretation of L-functions, using the Lefschetz trace formula as in Talk 3. Put together the results on étale cohomology we have proved so far to prove the rationality, functional equation and relationship to Betti cohomology of the zeta functions of smooth projective varieties. Reference: Sections 21, 26, 27 of [6].

Talk 5: The fundamental estimate. June 12 2017. Speaker: Joao L.
Present Chapter 3 of [2].

Talk 6: Lefschetz pencils; local theory. June 19 2017. Speaker: Zili Z.
Define Lefschetz pencils. Discuss the monodromy formalism, the Picard-Lefschetz formula and the local monodromy of Lefschetz pencils. References: Section 4 of [2], Chapter III of [5].

Talk 7: Lefschetz pencils; global theory. June 26 2017. Speaker: Giorgi V.
Present Section 5 of [2]. You can also use Chapter III of [5] as a reference.

Talk 8: Completion of the proof. July 10 2017. Speaker: Georg L.
Prove that the fundamental estimate from Talk 5 applies to the situation we are considering. Apply the theory of Lefschetz pencils to complete the proof of the Riemann hypothesis. Reference: Sections 6 and 7 of [2].

Talk 9: Alternative proof. July 17 2017. Speaker: Linus H.
Discuss Deligne's theory of weights and the alternative proof in Weil II. Reference: [3]

Talk 10: Applications. July 24 2017. Speaker: Ana C.
Sketch Deligne's proof of the Ramanujan-Petersson conjecture concerning the Fourier coefficients of modular forms of weight $k \geq 2$. Reference: [4]

References

[1] P. Deligne, *Cohomologie étale*, SGA 4 1/2, Lecture Notes in Math 569, Springer-Verlag.
[2] P. Deligne, *La conjecture de Weil, I*, Publ. Math. I.H.E.S 43, 273-307 (1974).
[3] P. Deligne, *La conjecture de Weil, II*, Publ. Math. I.H.E.S 52, 137252 (1980).
[4] P. Deligne, *Formes modulaires et reprsentations l-adiques*, Sminaire Bourbaki vol. 1968/69 Exposs 347-363, Lecture Notes in Mathematics, 179, Berlin, New York: Springer-Verlag.
[5] E. Freitag, R. Kiehl, *Étale cohomology and the Weil conjectures*, Springer-Verlag.
[6] J. Milne, *Lectures on étale cohomology.* Available at http://www.jmilne.org/math/CourseNotes/LEC.pdf .